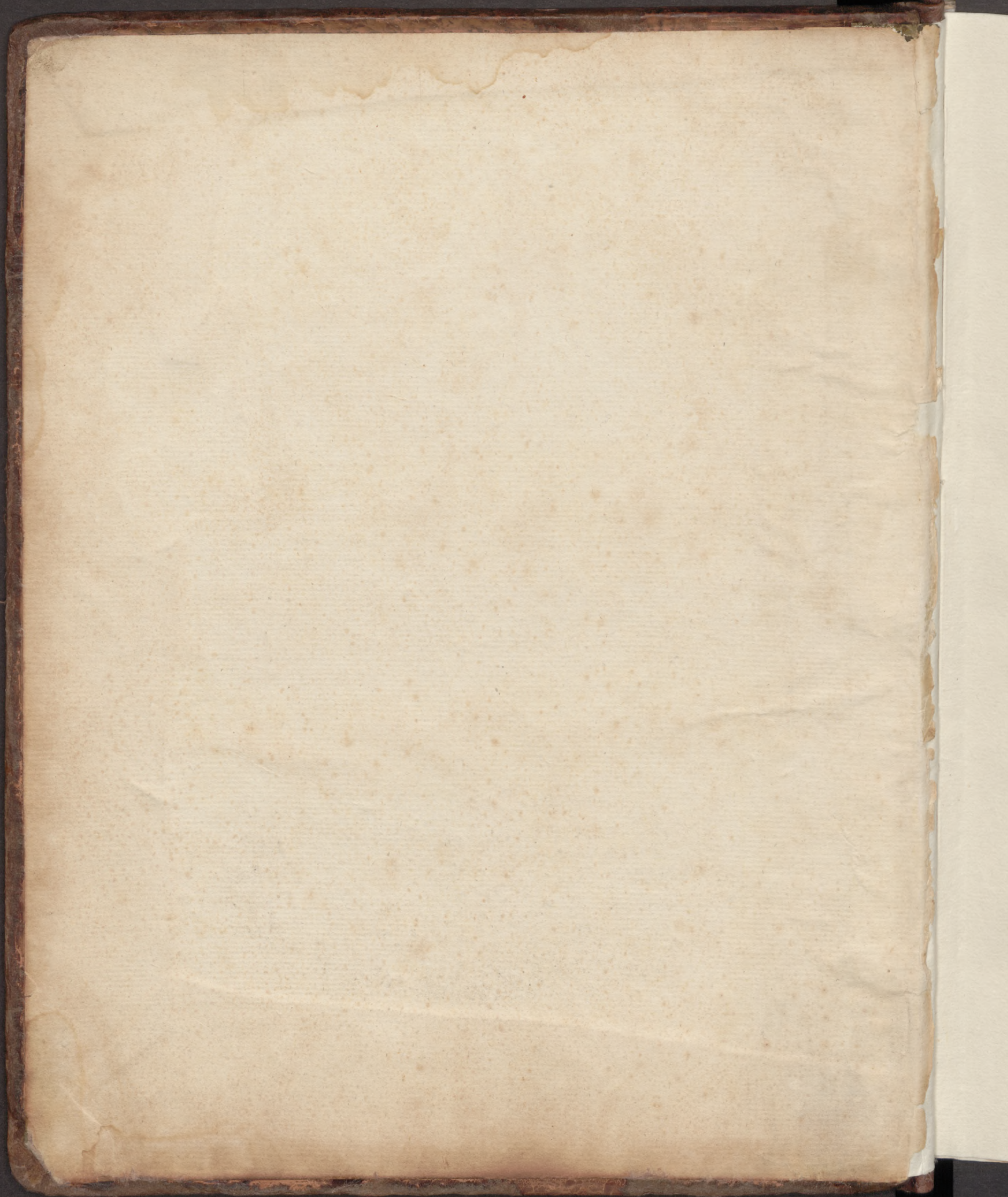
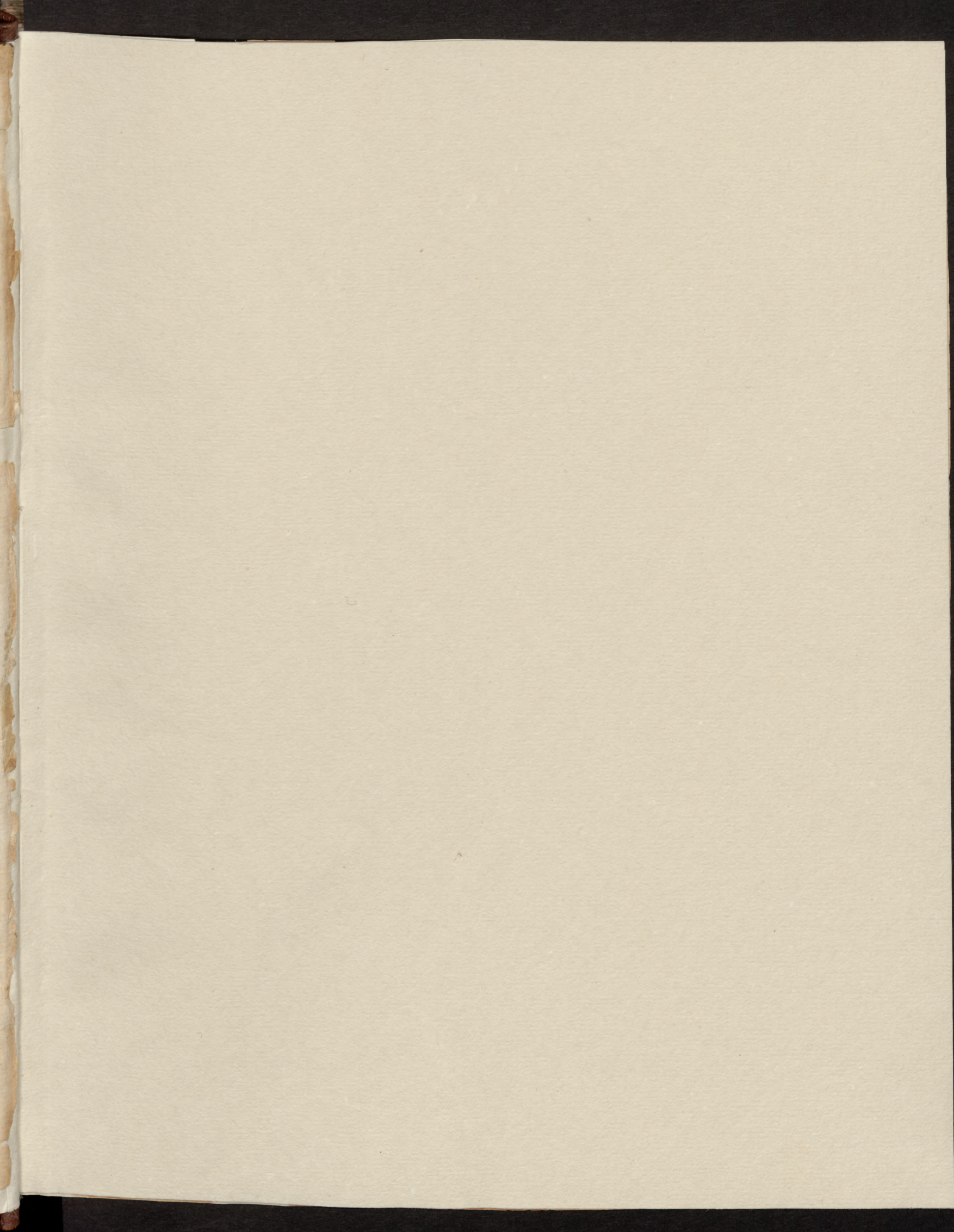
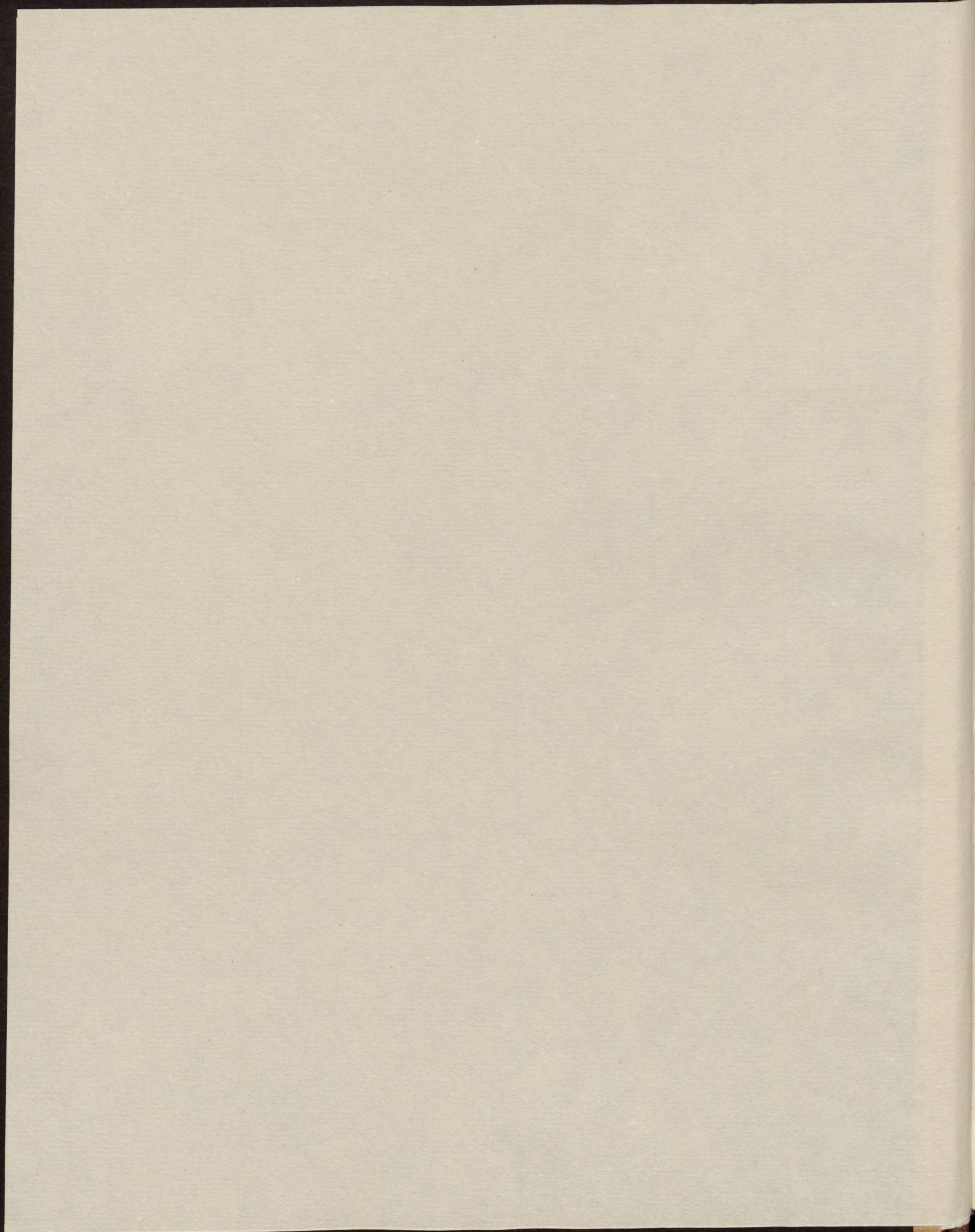


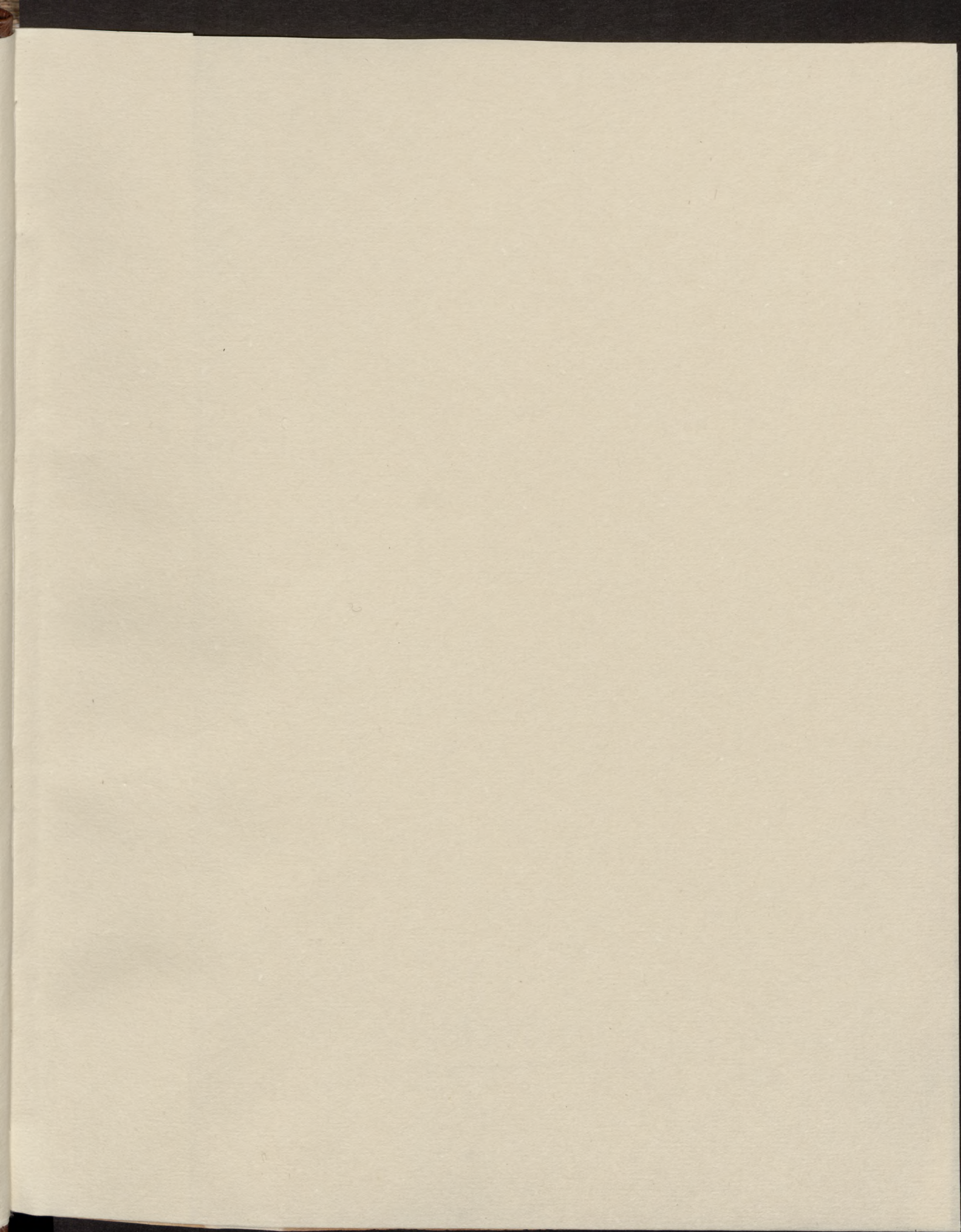
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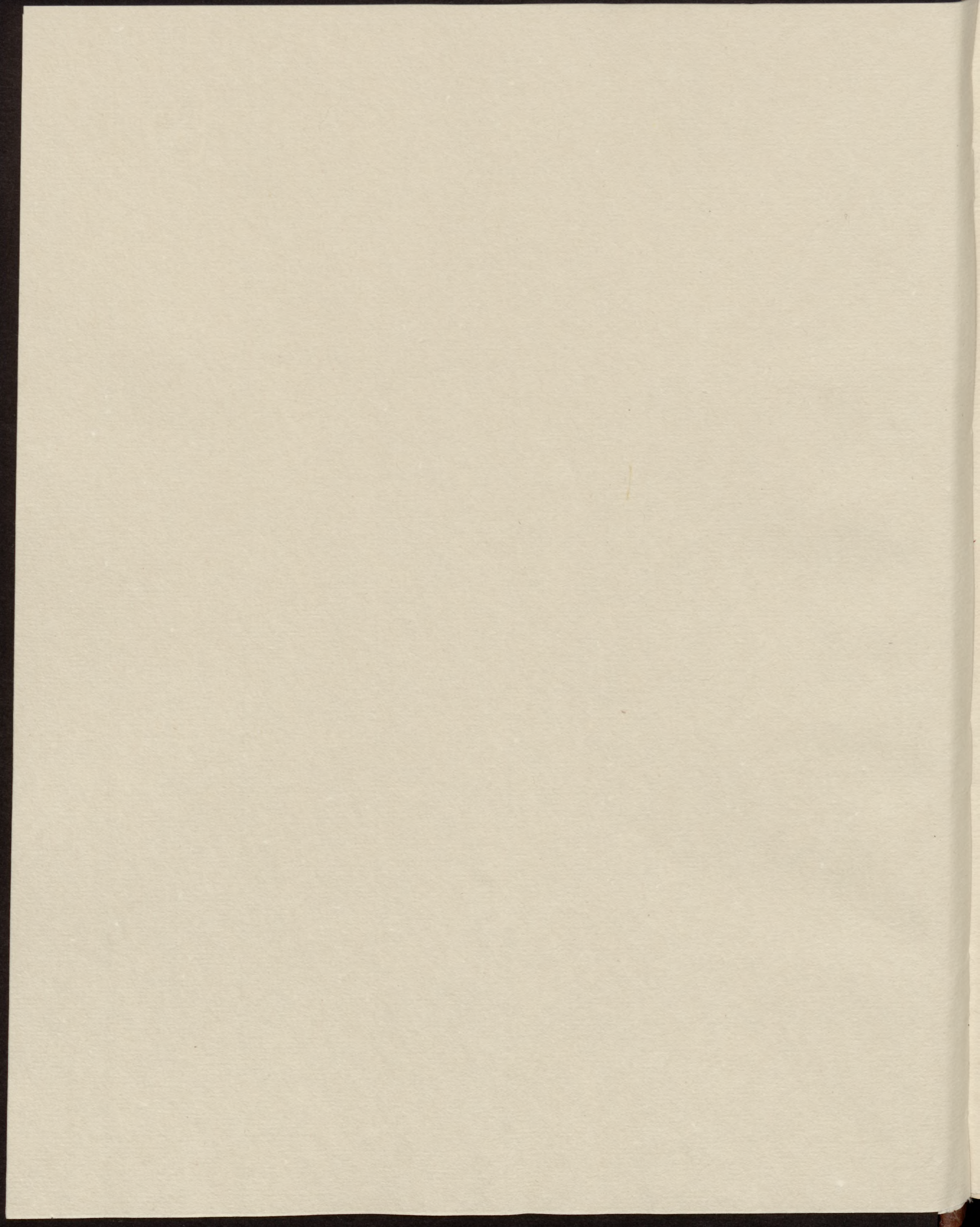
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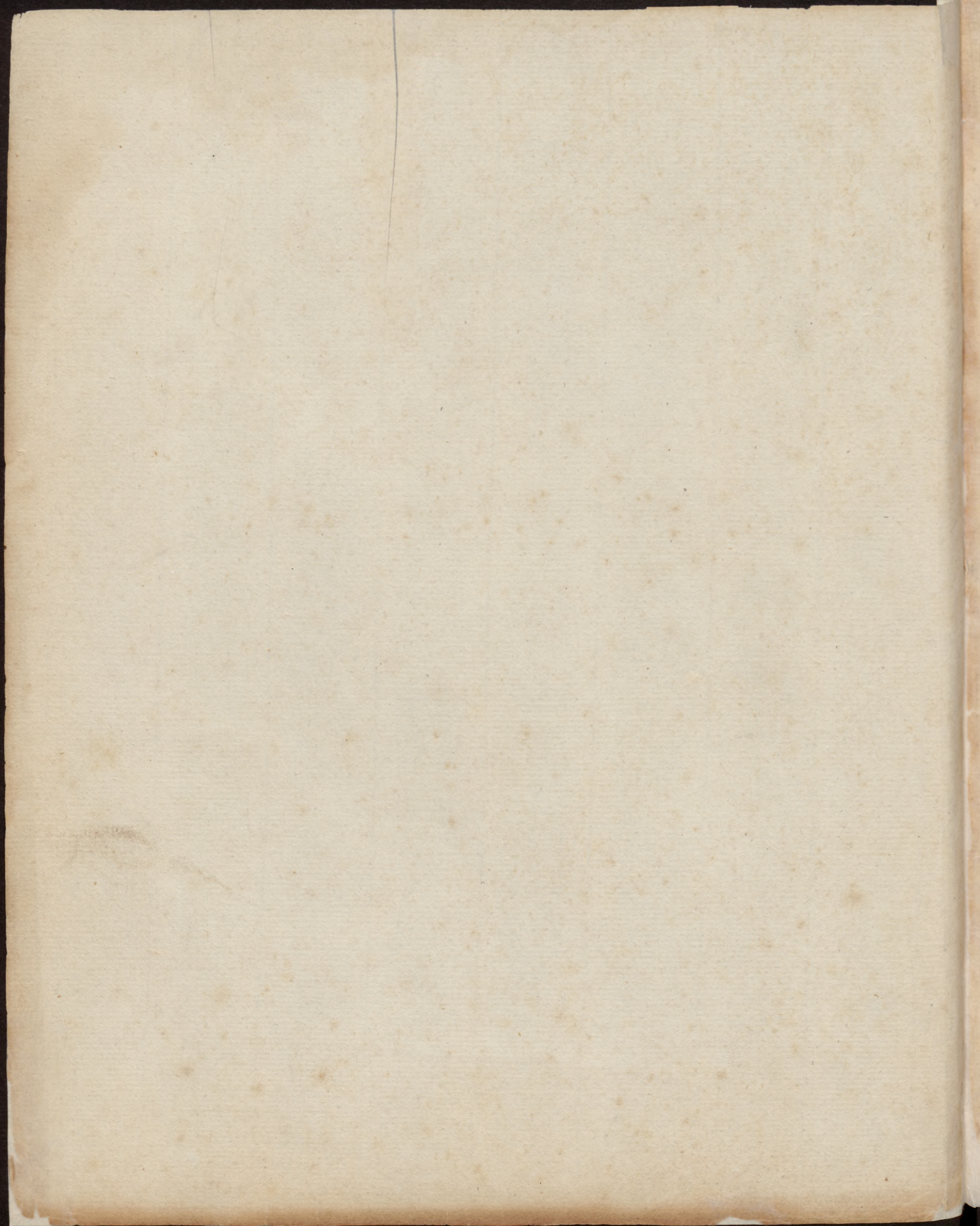












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the University

Died 19 Apr. 1739

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of Mathematicks in
of Cambridge

Aged 56

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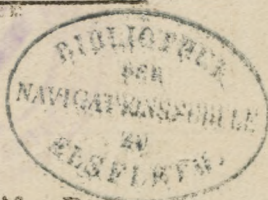
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IN TEN BOOKS:

BY NICHOLAS ANDERSON

An account of the AUTHOR'S LIFE AND CHARACTER



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ADVERTISEMENT.

THE learned Author of this Book having, in his paper of Proposals, drawn up a short Scheme of his Work, it is thought proper to reprint it here, in order to give the Reader in a few words an idea of his performance.

This Work (*says the Author*) is chiefly intended for the instruction of young beginners, and for the use of those who have such under their care. It is divided into ten books, with an introduction prefixed concerning vulgar and decimal fractions; the former being absolutely necessary to a thorough knowledge of Algebra. In this introduction, all the reductions and operations of fractions are fully and clearly accounted for, and such a fund of reasoning established thereby in the mind of the learner, as cannot fail to furnish him with proper rules to work by in all cases wherever fractions are concerned, without any further assistance.

The first book treats of the nature of Algebra and Algebraic quantities; of their addition, subtraction, multiplication and division; of proportion, fractions, and extraction of roots in Algebra; and in the last place, of the manner of resolving simple equations, illustrated by a considerable number of examples. In this book, under the head of multiplication is shewn, how by bare multiplication alone many useful theorems may be invented, and have been invented, both in Arithmetic and Geometry. Under the heads of division and extraction of roots, some account is given of the rise and continuation of infinite serieses, and how they may be tried by involution; but they are here touched upon only so far as may be apprehended by the meanest capacities, all other considerations concerning them being reserved to other parts of this treatise, where it may reasonably be presumed, the learner will be better prepared for them.

The second book contains a great variety of entertaining questions producing simple equations, and solved partly by single positions, and partly by more; where all the useful methods of extermination are explained.

The third book treats of quadratic equations, and of the manner of resolving them, exemplified in various questions introduced for that purpose; where throughout the whole are occasionally interspersed many curious observations concerning the roots of equations, both possible and impossible.

The fourth book treats of pure Algebra, that is, where letters of the alphabet are used, not only to represent unknown quantities, but also such as are known, which in tenderness to the learner has hitherto been avoided. Here several of the former problems are resumed, more indefinitely

finitely proposed, and general solutions given them by general theorems or canons, first traced out analytically, and afterwards demonstrated synthetically, whereby the learner may make himself thorough master of both sorts of demonstrations.

The fifth book gives the solutions of many curious questions of that sort which admit of more answers than one, and some even of an infinite number, by a general method very easy to be comprehended, and (as the author conceives) entirely new. Here are demonstrated many elegant and useful theorems relating both to whole numbers and fractions; particularly that of Mr. Cotes, for finding the least numbers that will express a given ratio to any given degree of exactness. Here also occasion is taken to introduce the reader into an acquaintance with the most entertaining parts of *Euclid's* doctrine of Incommensurables, where he will meet with more subtil and more refined reasoning than perhaps in any other part of the Mathematics whatever; and all levelled to the lowest capacity.

The sixth book is a choice collection of such questions as are usually known by the name of *Diophantine questions*; the solutions of them here given are very easy and intelligible, and in some cases perhaps may be thought preferable to any solutions of the same problems in *Diophantus*, or in any of his commentators or followers: for as *Diophantus's* porisms are entirely lost, he lies, in many cases, at the mercy of his commentators.

The seventh book treats in the first place of the doctrine of proportion as it is delivered in the fifth book of the Elements. Here it is shewn that the common idea of proportionality may, without the least disguise, be so enlarged, as to extend to Incommensurables: and having thus established an infallible and adequate criterion of proportionality, the fifth and seventh definitions to the fifth book of the Elements are shewn to be no more than plain and natural consequences of it, but more proper upon many accounts for *Euclid's* purpose, in carrying on his system of Geometry. Then all the propositions of the fifth book of the elements are clearly and succinctly demonstrated in their order, and as near as possible, in *Euclid's* manner, so as to lose nothing either of the force or elegance of his demonstrations; and yet such an easy and familiar turn is given to the whole, that it is to be hoped this book, which has always been looked upon as an unfurmountable rub in the way through the six first books of the Elements, will now be read with as much ease to the imagination, nay more, than any other part of the Elements whatever. The latter part of this book gives a clear and distinct account of the composition and resolution of ratios, and of their great use in Natural and Mechanical Philosophy: insomuch that it is to be hoped, this part of the doctrine of proportion will be no longer a mystery to any one who will read it with the least degree of attention.

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The eighth book applies Algebra to Geometry, and by the help of a few plain and easy problems, conveys to the mind of the learner the most sublime mysteries of that science. Here the composition of the Geometrical problems is first deduced from the *analysis*, and then synthetical demonstrations are formed from the constructions there given, without any regard to the *analysis*. The second part of this book contains the doctrine of solids, so far as it relates to prisms, cylinders, pyramids, cones, spheres, spheroids, &c; the principal properties whereof are taken out of *Euclid*, *Archimedes* and others, and demonstrated after the simplest manner, without the least stress laid upon the imagination.

The ninth book consists of several miscellaneous tracts; as first, of powers and their indexes; 2dly, of *Newton's* method of evolving a binomial, considered in it's full extent; 3dly, of logarithms, their nature and use, and particularly of *Briggs's* logarithms; 4thly, of logarithmotechny, or the method of computing logarithms, drawn from their simplest properties; 5thly, of *Newton's* invention of divisors; and 6thly, of the Arithmetic of surd quantities.

The tenth and last book treats first of equations in general, and then of cubic and biquadratic equations in particular, gives the best methods of resolving them where they will admit of an accurate resolution, and proper rules for approximations where they will not; particularly *Newton's* method is here described and explained.

By the account here given it is easy to see, that the author intended this treatise, not as a course of Algebra only, but also to promote, as far as possible, the study of Geometry, by removing or explaining all those difficulties which, from a long experience, he knows are apt to retard, if not discourage, young students in their progress through the Elements.

Though great care hath been taken to give the publick a correct edition of this excellent work, yet there are still remaining several errors of the press, though most of them are of little consequence. If the candid reader will take the trouble to correct the most material of those that are mentioned in the catalogue of errata, I hope and believe, he will find no others that do in the least obscure the sense of the Author.

*There is one passage in the ninth book, relating to some matters of fact, wherein I conceive, the learned Author was mistaken as to some few particulars; concerning which I had not an opportunity to discourse with him, because of his sudden and unexpected sickness and death. The passage is in page 620, where the Author having recommended Mr. Briggs's system of logarithms as the best accommodated for practice which are now in use, proceeds in the following words: The Lord Napier, a Scotch Nobleman, was the first inventor of logarithms: but our countryman Mr. Briggs, Professor of Geometry in Gresham College, was undoubtedly the first who thought of
this*

this system, and proposing it to the noble inventor, the Lord Napeir, he afterwards published it with that Lord's consent and approbation.

Now in the first place I observe, that our late worthy Professor wrote the name of the celebrated inventor of logarithms differently in different parts of his manuscript copy. Sometimes he is called Nepier, and sometimes Neper. I have an english book written by this illustrious author, and printed in the year 1611, which in the title page is said to be set forth by John Napeir L. of Marchiston: accordingly I have every where in the following work, caused his name to be printed Napeir.

Secondly, whereas our author styles him The Lord Napeir, a Scotch Nobleman; this, I think, ought to be understood in a qualified sense. In The Peerage of Scotland written by George Crawford Esq; and printed at Edinburgh in 1716, I find, page 364, that Sir Archibald Napier, eldest son to the foresaid John, was the first of this family that was created a Peer of Scotland, being raised to that honour by King Charles the first in the year 1627, his father (the inventor of logarithms) having departed this life several years before, in the reign of King James the sixth. I confess, Mr. Briggs writes him Johannes Neperus Baro Merchistonii: but if the account given of this family by Mr. Crawford be true, the inventor of logarithms can only be supposed to have been one of the inferior order of Barons in Scotland, and not a Peer of that Realm.

Thirdly, whereas Dr. Saunderson styles Mr. Briggs, Professor of Geometry in Gresham College; it is true, he was so, when Napeir first published his Canon mirificus logarithmorum: but when Mr. Briggs published his own Arithmetica logarithmica in 1624, he was Savilian Professor of Geometry at Oxford, as appears from the title page of that book.

Fourthly, whereas Mr. Professor Saunderson saith, that our countryman Briggs was undoubtedly the first who thought upon this system of logarithms which is now in use; I know not by what authority he asserts this. On the contrary, Dr. Keill in the preface to his treatise de logarithmis writes thus: Aliam deinde magis commodam logarithmorum formam Neperus excogitavit, et communicato consilio cum Domino Henrico Briggio, Geometriae in Academia Oxoniensi Professore, hunc socium operis sibi adjunxit, ut logarithmos in meliorem formam redactos completeret. See likewise Briggs's preface to his Arithmetica logarithmica.

I have nothing further to add, but that the Reader is obliged to the ingenious Mr. Abraham de Moivre for two excellent performances inserted in this work: the former at the end of the fourth book, is his curious solution of two problems concerning proportionals; and the latter in an appendix to these Elements, is a noble discovery of a rule for extracting the cubick, or any other root of (what is called) an impossible binomial, such as $a + \sqrt{-b}$, and also for extracting any root out of a given power thereof.

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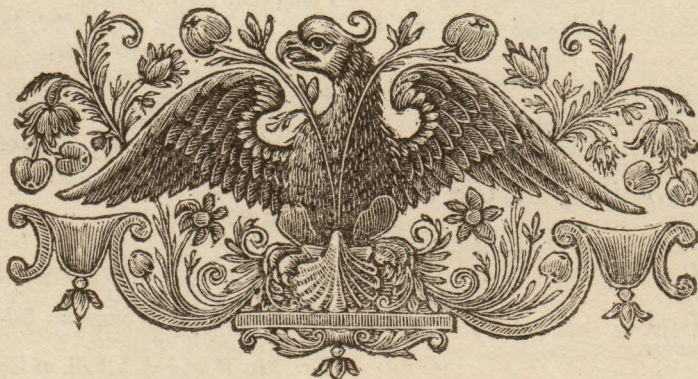
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ERRATA.

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*Note, The most material errors are marked with *.*

- * Pag. 4. in the answer to the tenth question, for 15625 feet read 15625 square feet.
- Pag. 45. lin. 8. read as any. And lin. 22, 23. read decimal.
- Pag. 57. lin. penult. read the two first products.
- Pag. 64. lin. 6 from the bottom, read it was shewn.
- * Pag. 138. lin. 12. for substitute read substituting.
- * Pag. 239. lin. 4 from the bottom, for $s \times \frac{q}{q-r} = \frac{qs}{q-r}$ read $s \times \frac{q}{q-r} = \frac{qs}{q-r}$.
- * And in the last line, for $\frac{q}{q-r}$ read $\frac{q}{q-r}$.

In the Second Volume.

- Pag. 390. lin. 11. *dele* must.
- * Pag. 536. lin. 14. for $x^2 - a^2x^2$ read $x^4 - a^2x^2$.
- Pag. 551. lin. 21. for where read when.
- Pag. 604. lin. ult. for x^{-3} read x^3 .
- Pag. 658. lin. 4. read DEFINITIONS.
- * Pag. 671. lin. 3. for binomial read binomial root.
- Pag. 683. line 21. read left.
- * Pag. 703. in the first line of the sixth example, for $\frac{q}{4}$ read $\frac{qq}{4}$.
- Pag. 738. line 13. *dele* a.

In the Table of the Contents.

- * Art. 231. for two numbers read two square numbers.
- Art. 403. for Napeirs read Napier's.

Errata in the Pointing.

- Pag. 44. lin. 16. after .4375 put a comma.
- Pag. 81. lin. 18. at the end of the line, after 2aa, instead of a comma put a semicolon.
- Pag. 91. at the end of the fifth example, put a full stop instead of a comma.
- Pag. 124. lin. 8 from the bottom, at the end of the line, put a semicolon instead of a comma.
- Pag. 167. lin. 25. after no square number put a semicolon instead of a comma.
- Pag. 178. line the last but one of the second example, after ± 1 put a semicolon.
- Pag. 199. lin. 16. after $x = \frac{32400 - 675x}{x}$ put a semicolon.
- Pag. 238. lin. 7. put a full stop at the end of the paragraph.
- Pag. 331. lin. 5. at the end of the paragraph, put a full stop instead of a comma.

In the Second Volume.

- Pag. 542. lin. 5 from the bottom, for DEG; but read DEG. But.
- Pag. 668. lin. 16. for root: and vice versa read root. And vice versa.
- Pag. 738. lin. antepen. at the end of the line, after $xx + 3x + 2 = 0$ put a comma.

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but -4 multiplied into $+3$, or $+4$ into -3 produces in either case -12 .

If the reader expects a demonstration of this rule, he must first be advertised of two things: *first*, that numbers are said to be in arithmetical progression, when they increase or decrease with equal differences, as 0, 2, 4, 6; or 6, 4, 2, 0; also as 3, 0, -3 ; 4, 0, -4 ; 12, 0, -12 ; or -12 , 0, $+12$: whence it follows, that three terms are the fewest that can form an arithmetical progression; and that of these, if the two first terms be known, the third will easily be had: thus if the two first terms be 4 and 2, the next will be 0; if the two first be 12 and 0, the next will be -12 ; if the two first be -12 and 0, the next will be $+12$, &c.

2dly, If a set of numbers in arithmetical progression, as 3, 2 and 1, be successively multiplied into one common multiplicator, as 4, or if a single number, as 4, be successively multiplied into a set of numbers in arithmetical progression, as 3, 2 and 1, the products 12, 8 and 4, in either case, will be in arithmetical progression.

This being allowed, (which is in a manner self-evident,) the rule to be demonstrated resolves itself into four cases:

1st, That $+4$ multiplied into $+3$ produces $+12$.

2dly, That -4 multiplied into $+3$ produces -12 .

3dly, That $+4$ multiplied into -3 produces -12 .

And *lastly*, that -4 multiplied into -3 produces $+12$. These cases are generally expressed in short thus: first $+$ into $+$ gives $+$; secondly $-$ into $+$ gives $-$; thirdly $+$ into $-$ gives $-$; fourthly $-$ into $-$ gives $+$.

Case 1st. That $+4$ multiplied into $+3$ produces $+12$, is self-evident, and needs no demonstration; or if it wanted one, it might receive it from the first paragraph of the 3d article; for to multiply $+4$ by $+3$ is the same thing as to add $4 + 4 + 4$ into one sum; but $4 + 4 + 4$ added into one sum give $+12$, therefore $+4$ multiplied into $+3$, gives $+12$.

Case 2d. And from the second paragraph of the 3d art. it might in like manner be demonstrated, that -4 multiplied into $+3$ produces -12 : but I shall here demonstrate it another way, thus: multiply the terms of this arithmetical progression 4, 0, -4 , into $+3$, and the products will be in arithmetical progression, as above; but the two first products are 12 and 0; therefore the third will be -12 ; therefore -4 multiplied into $+3$, produces -12 .

Case 3d. To prove that $+4$ multiplied into -3 produces -12 ; multiply $+4$ into $+3$, 0, and -3 successively, and the products will be in arithmetical progression; but the two first products are 12 and 0, there-

H

fore

two first /

fore the third will be -12 ; therefore $+4$ multiplied into -3 produces -12 .

Case 4th. Lastly, to demonstrate, that -4 multiplied into -3 produces $+12$, multiply -4 into 3 , 0 , and -3 successively, and the products will be in arithmetical progression; but the two first products are -12 and 0 , by the second case; therefore the third product will be $+12$; therefore -4 multiplied into -3 produces $+12$.

$$\begin{array}{r} \text{Cas. 2d. } +4, \quad 0, \quad -4 \\ \quad +3, \quad +3, \quad +3 \\ \hline \quad +12, \quad 0, \quad -12. \end{array}$$

$$\begin{array}{r} \text{Cas. 3d. } +4, \quad +4, \quad +4 \\ \quad +3, \quad 0, \quad -3 \\ \hline \quad +12, \quad 0, \quad -12. \end{array}$$

$$\begin{array}{r} \text{Cas. 4th. } -4, \quad -4, \quad -4 \\ \quad +3, \quad 0, \quad -3 \\ \hline \quad -12, \quad 0, \quad +12. \end{array}$$

These 4 cases may be also more briefly demonstrated thus: $+4$ multiplied into $+3$ produces $+12$; therefore -4 into $+3$, or $+4$ into -3 ought to produce something contrary to $+12$, that is, -12 : but if -4 multiplied into $+3$ produces -12 , then -4 multiplied into -3 ought to produce something contrary to -12 , that is, $+12$; so that this last case, so very formidable to young beginners, appears at last to amount to no more than a common principle in Grammar, to wit, that two negatives make an affirmative; which is undoubtedly true in Grammar, though perhaps it may not always be observed in languages.

Of the multiplication of simple algebraic quantities.

6. These things premised, the multiplication of simple algebraic quantities is performed, first by multiplying the numeral coefficients together, and then putting down, after the product, all the letters in both factors, the sign (when occasion requires) being prefixed as above directed. Thus $4b$ multiplied into $3a$ produces $12ab$.

Though this kind of language (for it is no more) like all others, be purely arbitrary, yet that a more rational one could not have been invented for this purpose, will appear by the following consideration. If any quantity, as b , is to be multiplied by any number, as 2 , 3 , or 4 , the product cannot be better represented than by $2b$, $3b$, $4b$, &c; therefore if b is to be multiplied by a , the product ought to be called ab : but if b multiplied into a produces ab , then $4b$ multiplied into a ought to produce 4 times as much, that is, $4ab$; lastly, if $4b$ multiplied into

a produces $4ab$, then $4b$ multiplied into $3a$ ought to produce 3 times as much, that is, $12ab$.

Hence it is, that whenever in Algebra two or more letters are found together, as they stand in a word without any thing between them, they signify the product arising from a continual multiplication of the quantities represented by them: thus ab signifies the product of a and b multiplied together; and abc signifies the product of the quantity ab multiplied into c : thus aa signifies the product of a multiplied into itself, or the square of a , and not $2a$; and therefore whoever shews himself unable to distinguish betwixt $2a$ and aa , discovers as great a weakness as one that is not able to distinguish betwixt 2 dozen and a dozen dozen or 12 times 12.

It is a matter of no great consequence in what order the letters are placed in a product; for ab and ba differ no more from one another than 3 times 4, and 4 times 3: and yet it is convenient that a method be observed, lest like quantities be sometimes taken for unlike; therefore the best way will be, to give those letters the precedency in a product, that have it in the alphabet; except when an unknown quantity is multiplied by some known one, and then it is usual to place the known quantity before it.

Note. For the signification of this mark, \times see introduct. at the close of the 7th article. Note also, that this mark $=$ is a mark of equality, shewing that the quantities between which it stands, are equal to each other, and must be read as the sense requires: thus $2 \times 6 = 3 \times 4 = 12$ may be read thus; 2×6 equals 3×4 equal to 12: or thus; 2×6 is equal to 3×4 , which is equal to 12.

Examples of simple algebraic multiplication.

- | | |
|----------------------------------|---|
| 1st, $4ab \times 5a = 20aab.$ | 2d, $-5ab \times 6bc = -30abbc.$ |
| 3d, $6ac \times -7bd = -42abcd.$ | 4th, $-7a \times -b = +7ab.$ |
| 5th, $x \times 3x = 3xx.$ | 6th, $-x \times -x = +xx.$ |
| 7th, $-5ab \times +3 = -15ab.$ | 8th, $\frac{2}{3}a \times \frac{4}{5}b = \frac{8}{15}ab.$ |

Distinctions to be observed betwixt addition and multiplication.

That the young algebrist may not confound the operations of addition and multiplication, as is frequently done; I shall here set down some marks of distinction, which he ought to attend to:

As *first*, a added to a makes $2a$, but a multiplied into a makes aa .
2dly, a added to 0 makes a , but a multiplied into 0 makes 0 .
3dly, a added to $-a$ makes 0 , but a multiplied into $-a$ makes $-aa$.
4thly, $-a$ added to $-a$ makes $-2a$, but $-a$ multiplied into $-a$ makes $+aa$.
5thly, a added to 1 makes $a+1$, but a multiplied into 1 makes a .
6thly, $2a$ added to $-3b$ makes $2a-3b$, but $2a$ multiplied into $-3b$ makes $-6ab$.

For a further confirmation of the learner, I have added, by way of exercise in his algebraic language, the following equations; which I desire he would compute after me. Suppose $a=7$, and $b=3$: then we shall have *1st*, $a+b=10$. *2dly*, $a-b=4$. *3dly*, $4a+5b=43$. *4thly*, $4a-5b=13$. *5thly*, $aa=49$. *6thly*, $ab=21$. *7thly*, $bb=9$. *8thly*, $aaa=343$. *9thly*, $aab=147$. *10thly*, $abb=63$. *11thly*, $bbb=27$. *12thly*, $aa+2ab+bb=49+42+9=100$. *13thly*, $aa-2ab+bb=49-42+9=16$. *14thly*, $aaa+3aab+3abb+bbb=343+441+189+27=1000$. *15thly*, $aaa-3aab+3abb-bbb=343-441+189-27=64$.

Of powers and their indexes.

7. Whenever in multiplication a letter is to be repeated oftener than once, it is usual by way of compendium, to write down the letter with a small figure after it, shewing how often that letter is to be repeated: thus instead of xx we write x^2 , instead of xxx we write x^3 , instead of $xxxx$ we write x^4 , &c. These products are called powers of x ; the figures representing the number of repetitions, are called the indexes of those powers; and the quantity x from whence all these powers arise, is called the root of these powers, or the first power of x ; x^2 is called the second power of x , x^3 the third power, x^4 the fourth power, &c. *Vieta*, *Oughtred*, and some other analysts, instead of small letters used capitals, and instead of numeral indexes, distinguished these powers by names: thus *Vieta* in particular, called x^2 , *Xsquare*; x^3 , *Xcube*; x^4 , *Xsquare-square*; x^5 , *Xsquare-cube*; x^6 , *Xcube-cube*; x^7 , *Xsquare-square-cube*, &c: which names *Oughtred* contracted, and wrote them thus; *Xq*, *Xc*, *Xqq*, *Xqc*, *Xcc*, *Xqqc*, &c. but now these names are pretty much out of use, except the two first, when applied to a line squared or cubed.

If we suppose $x=5$, we shall have $2x=10$, $x^2=25$, $3x=15$, $x^3=125$, $4x=20$, $x^4=625$, &c.

The multiplication of these powers is easy: thus $x^2 \times x^3 = x^5$, because $xxxxxx = xxxxxx$: whence it may be observed, that the addition of indexes will always answer to the multiplication of powers, provided they be powers of the same quantity; for as $2+3=5$, so $x^2 \times x^3 = x^5$, &c:
 but

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but if they be powers of different quantities, their indexes must not be added: thus $a^2 \times x^3 = a^2 x^3$, and $a^2 x^3 \times a^4 x^5 = a^6 x^8$. And here it must be observed, that if a number be found between two letters, it must always be referred to the former letter: thus $a^2 x^3$ does not signifie $a \times 2 x^3$, but $a^2 \times x^3$.

The multiplication of surds.

8. This mark $\sqrt{}$ signifies the square root of the number to which it is prefixed, and is generally prefixed to numbers whose square root cannot be otherwise expressed, either by whole numbers or fractions: thus $\sqrt{2}$ signifies the square root of 2; \sqrt{a} the square root of a , &c. These roots are commonly called surd roots, or irrational roots, because their proportion to unity cannot be expressed in numbers.

Whenever two surd numbers are to be multiplied together, the shortest way will be, to multiply the numbers themselves one into the other without any regard to the radical sign, and then to prefix the radical sign to the product. Thus if \sqrt{a} is to be multiplied into \sqrt{b} , the product will be \sqrt{ab} ; which I thus demonstrate: let $\sqrt{a} = x$, and $\sqrt{b} = y$; then will $x^2 = a$, and $y^2 = b$, and $x^2 y^2 = ab$, and $xy = \sqrt{ab}$; but xy , or $x \times y = \sqrt{a} \times \sqrt{b}$ by the supposition; therefore, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. Thus $\sqrt{2} \times \sqrt{3} = \sqrt{6}$.

These multiplications are of considerable use not only in matters of speculation, but also in practice: for suppose I had occasion to multiply the square root of 2 into the square root of 3, if I had not this rule, I must first extract the root of 2, to what degree of exactness I think proper for my purpose; then again I must extract the root of 3 to the same degree of exactness; and lastly I must multiply these two roots together, before I can obtain the number wanted: but after it is known that $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, the whole operation will then be reduced to the extraction of the root of 6 only: nay it sometimes happens, that two roots, though both irrational, shall have a rational product: thus $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$, and $\sqrt{ab^2} \times \sqrt{ac^2} = \sqrt{a^2 b^2 c^2} = abc$.

Of the multiplication of compound algebraic quantities.

9. The multiplication of compound algebraic quantities is performed, first by multiplying the multiplicand into every particular member of the multiplier, and then reducing the whole product into the least compass possible.

As for example; let it be required to multiply this compound quantity $6x - 7a - 8b$ into this compound quantity $2x - 3a + 4b$: here
having

having put down the multiplicand, and the multiplier under it, and beginning at the left hand, (for it is all one which way the operation is carried on,) I multiply the whole multiplicand into $2x$, the first member of my multiplier, and the product is $12xx - 14ax - 16bx$, which I put down: then I multiply the multiplicand into $-3a$, the next member of the multiplier, and the product is $-18ax + 21aa + 24ab$; whereof the first member $-18ax$, I place under $-14ax$ before found, being of the same denomination, for the conveniency of adding; the rest, to wit, $+21aa + 24ab$, I place in the first line: this done, I now multiply by $4b$, the last member of the multiplier, and the product is $24bx - 28ab - 32bb$; whereof I place $24bx$ under $-16bx$, and $-28ab$ under $+24ab$, and the last member $-32bb$ I place in the first line, as having no quantity of the same denomination to join with it: lastly I reduce the whole product into the least compass possible; and it stands thus: $12xx - 32ax + 8bx + 21aa - 4ab - 32bb$. See the work:

$$\begin{array}{r}
 6x - 7a - 8b \\
 2x - 3a + 4b \\
 \hline
 12xx - 14ax - 16bx + 21aa + 24ab - 32bb \\
 \quad - 18ax + 24bx \qquad - 28ab \\
 \hline
 \text{Sum } 12xx - 32ax + 8bx + 21aa - 4ab - 32bb.
 \end{array}$$

Example 2d.

$$\begin{array}{r}
 3x + 4a - 5b \\
 3x - 4a + 5b \\
 \hline
 9xx + 12ax - 15bx - 16aa + 20ab - 25bb \\
 \quad - 12ax + 15bx \qquad + 20ab \\
 \hline
 9xx \quad * \qquad * \quad - 16aa + 40ab - 25bb.
 \end{array}$$

Example 3d.

$$\begin{array}{r}
 6xx - 7ax + 8aa \\
 2xx - 3ax + 4aa \\
 \hline
 12x^4 - 14ax^3 + 16a^2x^2 - 24a^3x + 32a^4 \\
 \quad - 18ax^3 + 21a^2x^2 - 28a^3x \\
 \qquad + 24a^2x^2 \\
 \hline
 12x^4 - 32ax^3 + 61a^2x^2 - 52a^3x + 32a^4.
 \end{array}$$

Example

Example 4th.

$$\begin{array}{r} a+b \\ a-b \\ \hline aa+ab-bb \\ -ab \\ \hline aa \quad * -bb. \end{array}$$

Example 5th.

$$\begin{array}{r} a+b \\ a+b \\ \hline aa+ab+bb \\ +ab \\ \hline aa+2ab+bb. \end{array}$$

Example 6th.

$$\begin{array}{r} a-b \\ a-b \\ \hline aa-ab+bb \\ -ab \\ \hline aa-2ab+bb. \end{array}$$

N. B. A dash over two or more quantities, signifies that all those quantities are to be taken into one conception, or to be considered as making up but one compound quantity : thus $\overline{a+b \times c} - d$ does not signify that which arises from multiplying $b \times c$, and then adding $a - d$ to the product, as it might be mistaken without the dash ; but it signifies the product of the whole quantity $\overline{a+b}$ multiplied into the whole quantity $\overline{c-d}$.

The proof of compound multiplication.

10. In the third example we multiplied $\overline{6xx-7ax+8aa}$ into $\overline{2xx-3ax+4aa}$, and the product amounted to $12x^4-32ax^3+61a^2x^2-52a^3x+32a^4$: let us try this in numbers, and see how it will answer. In order to which, we may suppose a and x equal to any two numbers whatever, but the simplest way of trial will be to make a equal 1, and $x=1$; and then we shall have in the multiplicand $6xx=6$, $-7ax=-7$, and $+8aa=+8$, and $6-7+8=7$; therefore the multiplicand is 7 : again, in the multiplier we have $2xx=2$, $-3ax=-3$, $+4aa=+4$, and $2-3+4=3$; therefore the multiplier is 3 : and 7 the multiplicand, multiplied into 3 the multiplier, gives

21 for the product. Let us now examine the several parts of the product, as they are here represented in letters, and see whether they will amount to that number: $12x^4=12$, $-32ax^3=-32$, $+61a^2x^2=+61$, $-52a^3x=-52$, $+32a^4=+32$; and $12-32+61-52+32$ amount to just 21. This may serve as a proof to the work, though not a necessary one; for it is not impossible but there may be a consistency this way, and yet the work be false; but this will rarely happen, unless it be designed. But the work may still be confirmed by making $a=1$, and $x=-1$; for then the multiplicand will be $6+7+8=21$; and the multiplier $2+3+4=9$; and the product $12+32+61+52+32=189$, which is the same with the product of 21 the multiplicand, multiplied into 9 the multiplier.

How general theorems may be obtained by multiplication in Algebra.

11. From these algebraic multiplications are derived and demonstrated many very useful theorems in all the parts of Mathematics; whereof I shall just give the learner a taste, and then proceed to another subject.

In the fourth example of compound multiplication we found, that $a+b$ multiplied into $a-b$ produced $aa-bb$; whence I infer, that *The sum and difference of any two numbers multiplied together will give the difference of their squares, and vice versa*: for a and b will represent any two numbers at pleasure; $a+b$ their sum, $a-b$ their difference, and $aa-bb$ the difference of their squares: thus if we assume any two numbers whatever, suppose 7 and 3, the difference of their squares is $49-9$, or 40; and 10 their sum, multiplied into 4 their difference, makes also 40.

But here I am to give notice once for all, that instances in numbers serve well enough to illustrate a general theorem, but they must not by any means be looked upon as a proof of it; because a proposition may be true in some particular cases instanced in, and yet fail in others; but whenever a proposition is found to be true *in speciebus*, that is, in letters or symbols, it is a sufficient demonstration of it, because these are universal representations.

In the 5th example it was shewn, that $a+b$ multiplied into itself produced $aa+2ab+bb$; whence I infer, that *If a number be resolved into any two parts whatever, the square of the whole will be equal to the square of each part, and the double rectangle, or product of the multiplication of those parts, added together*: thus if the number 10 be resolved into 7 and 3; 100 the square of 10, the whole, will be equal to 49 the square of 7, and

Art. 11, 12. *To express the sides of a right-angled triangle in numbers.* 65 and 9 the square of 3, and 42 the double product of 7 and 3 multiplied together: for $49 + 9 + 42 = 100$.

In the 6th example we found, that $a - b$ multiplied into itself, produced $aa - 2ab + bb$; whence I infer, that *If from the sum of the squares of any two numbers, be subtracted the double product of those numbers, there will remain the square of their difference*: for $aa + bb$ is the sum of the squares of a and b , and $2ab$ is their double product, and $aa - 2ab + bb$ was found to be the square of $a - b$, that is, the square of the difference of a and b : thus in the numbers 7 and 3, the square of 7 is 49, the square of 3 is 9, and the sum of their squares is 58; and if from this be subtracted the double product 42, the remainder will be 16, the square of 4, that is, the square of the difference of the numbers 7 and 3.

These two last theorems are in substance the fourth and seventh propositions of the second book of *Euclid*.

How to express the three sides of a right-angled triangle in rational numbers.

12. From these two last theorems may be solved a problem of no small estimation among Algebrists; which is, to find three numbers, that shall represent the three sides of a right-angled triangle; or rather, to find as many sets as we please of such numbers; that is, in other words, *To find three numbers of such a nature, that the sum of the squares of two of them may be equal to the square of the third*. Now it is plain that these three numbers $aa - 2ab + bb$, and $4ab$, and $aa + 2ab + bb$ are such, that the two first added together, are equal to the third: it is certain also, that of these three numbers, the two extremes are square numbers, that is, are such as will admit of an exact square root; for $aa - 2ab + bb$, is the square of $a - b$; and $aa + 2ab + bb$ is the square of $a + b$; therefore, if the middle number $4ab$ was a square number like the rest, we should have 3 square numbers, whereof the two first added together would be equal to the third; and consequently, the roots of these three squares, would be three numbers that would answer the condition of the problem. But the middle number $4ab$ will be a square, if a and b be square numbers: for if we suppose $a = rr$, and $b = ss$, we shall have $4ab = 4rrss$, which is a square number whose root is $2rs$: but the root of the first square $aa - 2ab + bb$, was $a - b$, which in this case is $rr - ss$; and the root of the third square, $aa + 2ab + bb$ was $a + b$, which is $rr + ss$; therefore these three numbers, $rr - ss$, $2rs$ and $rr + ss$ are such, that the square of the first added to the square of the second will make the square of the third; and this will be the case,

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what-

whatever numbers r and s are made to stand for. But $rr - ss$ is the difference of the squares of r and s , and $2rs$ is the double product of their multiplication, and lastly $rr + ss$ is the sum of the squares of r and s . Since then we are at liberty to assume what numbers we please for r and s , the problem will receive the following solution: *Take any two numbers at pleasure, and from these two numbers thus taken, derive three others thus: take the difference of their squares, the double product of their multiplication, and the sum of their squares, and the three numbers thus found will answer the condition of the problem.* As for example, let the numbers taken be 2 and 1: now the difference of the squares of these numbers is $4 - 1 = 3$; the double product of their multiplication is $2 \times 2 \times 1 = 4$; and the sum of their squares is $4 + 1 = 5$; therefore the numbers 3, 4 and 5 are such as will answer the condition proposed: and so we find them; for $3 \times 3 + 4 \times 4 = 5 \times 5$, that is, $9 + 16 = 25$. Again, let the numbers assumed be 3 and 2; and the difference of their squares will be $9 - 4 = 5$, their double product $2 \times 3 \times 2 = 12$, and the sum of their squares $9 + 4 = 13$; therefore 5, 12 and 13 is another set of numbers that will answer the condition of the problem: for $5 \times 5 + 12 \times 12 = 13 \times 13$, that is, $25 + 144 = 169$. Lastly, let 4 and 1 be the numbers assumed; and the difference of their squares will be $16 - 1 = 15$, their double product $2 \times 4 \times 1 = 8$, and the sum of their squares $16 + 1 = 17$; therefore 8, 15 and 17, will also answer the condition of the problem: for $8 \times 8 + 15 \times 15 = 17 \times 17$, that is, $64 + 225 = 289$.

Of the division of simple algebraic quantities.

13. The division of simple algebraic quantities, where it is possible in integral terms, is performed, first by dividing the numeral coefficient of the dividend by the numeral coefficient of the divisor, and then putting down after the quotient, all the letters in the dividend, that are not in the divisor; the sign of the quotient in division being determined by those of the divisor and dividend, just in the same manner as the sign of the product in multiplication is determined by those of the multiplicator and multiplicand; that is, if the signs of the divisor and dividend be both alike, whether they be both affirmative, or both negative, the quotient will be affirmative, otherwise it will be negative: thus if the quantity $-12ab$ is divided by $-3a$, the quotient will be $+4b$; which I thus demonstrate: In all division whatever, the quotient ought to be such a quantity, as being multiplied by the divisor, will make the dividend; therefore, to enquire for the quotient in our case, is nothing else, but to enquire what number, or quantity, multiplied into $-3a$, the divisor,

vifor, will produce $-12ab$, the dividend. First then I ask, what sign multiplied into $-$, the sign of the divisor, will give $-$ the sign of the dividend, and the answer is $+$; therefore $+$ is the sign of the quotient: in the next place I enquire, what number multiplied into 3, the coefficient of the divisor, will give 12, the coefficient of the dividend, and the answer is 4; therefore 4 is the coefficient of the quotient: lastly I enquire, what letter multiplied into a , the letter of the divisor, will produce ab , the denominator, or literal part of the dividend, and the answer is b ; therefore b is the letter of the quotient: and thus at last we have the whole quotient, which is $+4b$. And this way of reasoning will carry the learner through all the other cases.

Examples of simple division in Algebra.

Example 1st, $4ab \overline{) 24abbc} (6bc.$
 2d, $+7 \overline{) -35ab} (-5ab.$
 3d, $-x \overline{) -3xx} (+3x.$
 4th, $-9ab \overline{) +72ab} (-8.$
 5th, $-4a^3 \overline{) -60a^8} (+15a^5.$
 6th, $4x^2 \overline{) 60x^9} (+15x^7.$
 7th, $+4a^3x^2 \overline{) -60a^8x^9} (-15a^5x^7.$
 8th, $b \overline{) \frac{3}{4}ab} (\frac{3}{4}a.$
 9th, $\frac{2}{3} \overline{) \frac{4}{5}b} (\frac{6}{5}b.$

Of the notation of algebraic fractions.

Whenever a division according to the foregoing method is found impossible, the quotient cannot be otherwise expressed than by a fraction, whose numerator is the dividend, and denominator the divisor; see the introduction, art. 13. As if it was required to divide a by b , which division is impossible according to the foregoing rule, the quotient must be expressed by this fraction $\frac{a}{b}$, which is usually read thus a by b , that is, a divided by b , or the quotient of a divided by b : for in Algebra the word *by*, is generally speaking, appropriated to division, as the word *into* is to multiplication.

If the numerator, or denominator, or both, be compound quantities, the respective fractions must be written thus; $\frac{a+b}{c}$, $\frac{a}{b+c}$, $\frac{a+b}{c-d}$.

If a division be partly possible according to the foregoing rules, and partly impossible, it must be pursued as far as it is possible, and the rest

must be represented by a fraction, as in common division: thus if $ad + bd + c$ was to be divided by d , the quotient would be $a + b + \frac{c}{d}$.

Of the division of compound algebraic quantities.

14. The division of compound algebraic quantities is performed, first, by ranging the several members both of the divisor and dividend according to the dimensions of some letter common to them both, and then proceeding as in common arithmetick.

N. B. A quantity is said to be disposed according to the dimensions of any letter in it, when the highest power of that letter is placed first, and the next in order, and so on, as in the following example; where both the divisor and dividend are ranged according to the dimensions of the letter x .

Of this compound division, take the following example: let it be required to divide this quantity $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by this quantity $2x - 3a$. Here, as my divisor consists of two members, I take the two first members of the dividend for a first dividual; then I divide the first member of the dividual by the first member of the divisor, to wit, $48x^3$ by $2x$, and the quotient is $24xx$, which I put down in the quotient: this done, I multiply the divisor $2x - 3a$ by the quotient $24xx$, and the product is $48x^3 - 72ax^2$, which I place under my first dividual $48x^3 - 76ax^2$, and then subtracting the former from the latter, I find the remainder to be $-4ax^2$; to this remainder I bring down the next place of my dividend, which is $-64a^2x$, and so have a second dividual, to wit, $-4ax^2 - 64a^2x$: here again I divide the first member of this dividual, by the first member of the divisor, viz. $-4ax^2$ by $2x$, and the quotient is $-2ax$, which I put down in the quotient; then multiplying the divisor $2x - 3a$, by this last quotient $-2ax$, the product is $-4ax^2 + 6a^2x$, which being subtracted from the second dividual $-4ax^2 - 64a^2x$ leaves $-70a^2x$ for a remainder; to this remainder I bring down $+105a^3$ the last place in the dividend, and so have a third dividual $-70a^2x + 105a^3$; the first member whereof divided by the first member of the divisor, quotes $-35aa$, which being put down in the quotient, and the divisor multiplied by it, the product is $-70a^2x + 105a^3$, which being subtracted from the last dividual leaves no remainder; so that the whole quotient at last amounts to $24xx - 2ax - 35aa$. See the work:

Example

Example 1st.

$$\begin{array}{r}
 2x-3a \quad 48x^3-76ax^2-64a^2x+105a^3 \quad (24x^2-2ax-35a^2) \\
 \underline{48x^3-72ax^2} \\
 * \quad -4ax^2-64a^2x \\
 \quad \underline{-4ax^2+6a^2x} \\
 * \quad -70a^2x+105a^3 \\
 \quad \underline{-70a^2x+105a^3} \\
 * \quad *
 \end{array}$$

Example 2d.

$$\begin{array}{r}
 4x-5a \quad 48x^3-76ax^2-64a^2x+105a^3 \quad (12x^2-4ax-21a^2) \\
 \underline{48x^3-60ax^2} \\
 * \quad -16ax^2-64a^2x \\
 \quad \underline{-16ax^2+20a^2x} \\
 * \quad -84a^2x+105a^3 \\
 \quad \underline{-84a^2x+105a^3} \\
 * \quad *
 \end{array}$$

Example 3d.

$$\begin{array}{r}
 6x+7a \quad 48x^3-76ax^2-64a^2x+105a^3 \quad (8x^2-22ax+15a^2) \\
 \underline{48x^3+56ax^2} \\
 * \quad -132ax^2-64a^2x \\
 \quad \underline{-132ax^2-154a^2x} \\
 * \quad +90a^2x+105a^3 \\
 \quad \underline{+90a^2x+105a^3} \\
 * \quad *
 \end{array}$$

Example

Example 4th.

$$\begin{array}{r}
 3x^2 - 4x + 5 \) \ 18x^4 - 45x^3 + 82x^2 - 67x + 40 \ (\ 6x^2 - 7x + 8 \\
 \underline{18x^4 - 24x^3 + 30x^2} \\
 * \quad -21x^3 + 52x^2 - 67x \\
 \quad \underline{-21x^3 + 28x^2 - 35x} \\
 * \quad +24x^2 - 32x + 40 \\
 \quad \underline{+24x^2 - 32x + 40} \\
 * \quad * \quad *
 \end{array}$$

This, as I take it, is the most intelligible way of working these divisions, as being the nearest of kin to common division; but it may be somewhat contracted, as will easily be seen by working the foregoing examples over again thus:

Example 1st.

$$\begin{array}{r}
 2x - 3a \) \ 48x^3 - 76ax^2 - 64a^2x + 105a^3 \ (\ 24x^2 - 2ax - 35aa \\
 \underline{48x^3 - 72ax^2} \\
 * \quad -4ax^2 \\
 \quad \underline{-4ax^2 + 6a^2x} \\
 \quad \quad -70a^2x \\
 \quad \quad \underline{-70a^2x + 105a^3} \\
 \quad \quad * \quad *
 \end{array}$$

Example 2d.

$$\begin{array}{r}
 4x - 5a \) \ 48x^3 - 76ax^2 - 64a^2x + 105a^3 \ (\ 12x^2 - 4ax - 21a^2 \\
 \underline{48x^3 - 60ax^2} \\
 * \quad -16ax^2 \\
 \quad \underline{-16ax^2 + 20a^2x} \\
 * \quad -84a^2x \\
 \quad \underline{-84a^2x + 105a^3} \\
 \quad \quad * \quad *
 \end{array}$$

Example

Example 3d.

$$\begin{array}{r} 6x+7a \overline{) 48x^3-76ax^2-64a^2x+105a^3} \quad (8x^2-22ax+15aa \\ \underline{48x^3+56ax^2} \\ * -132ax^2 \\ \underline{-132ax^2-154a^2x} \\ +90a^2x \\ \underline{+90a^2x+105a^3} \\ * * \end{array}$$

Example 4th.

$$\begin{array}{r} 3x^2-4x+5 \overline{) 18x^4-45x^3+82x^2-67x+40} \quad (6x^2-7x+8 \\ \underline{18x^4-24x^3+30x^2} \\ * -21x^3+52x^2 \\ \underline{-21x^3+28x^2-35x} \\ * +24x^2-32x \\ \underline{+24x^2-32x+40} \\ * * * \end{array}$$

This sort of division may be proved the same way as in common division, to wit, by multiplying the divisor into the quotient, or the quotient into the divisor, and adding the remainder, if there be any; for then, if the product, or sum, be equal to the dividend, the division is right, otherwise not: thus if instead of dividing $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$ as in the first example, we divide $48x^3-76ax^2-64a^2x+110a^3$ by $2x-3a$, the quotient will be the same as before, to wit, $24x^2-2ax-35a^2$, but then there will be a remainder of $5a^3$; therefore *e converso*, if the quotient $24x^2-2ax-35a^2$ be multiplied into the divisor $2x-3a$, the product together with the remainder $5a^3$ will make the dividend $48x^3-76ax^2-64a^2x+110a^3$. See the proof:

$$\begin{array}{r}
 24x^2 - 2ax - 35a^2 \\
 \underline{2x - 3a} \\
 48x^3 - 4ax^2 - 70a^2x + 105a^3 \\
 \underline{-72ax^2 + 6a^2x} \\
 48x^3 - 76ax^2 - 64a^2x + 105a^3 \\
 \underline{+5a^3} \\
 48x^3 - 76ax^2 - 64a^2x + 110a^3.
 \end{array}$$

If, when the divisor and dividend are placed according to the dimensions of some common letter, there be any places wanting, these may be supplied by stars: as if it was required to divide $16x^4 - 72a^2x^2 + 81a^4$ by $2x - 3a$; there are wanting two places in the dividend, to wit, the places where the third and simple powers of x are concerned: these therefore being supplied, the dividend will stand thus; $16x^4 * - 72a^2x^2 * + 81a^4$. See the work:

$$\begin{array}{r}
 2x-3a) 16x^4 \quad * \quad -72a^2x^2 * + 81a^4 \quad (8x^3 + 12ax^2 - 18a^2x - 27a^3 \\
 \underline{16x^4 - 24ax^3} \\
 * \quad + 24ax^3 \\
 \quad + 24ax^3 - 36a^2x^2 \\
 \underline{\quad \quad \quad} \\
 * \quad - 36a^2x^2 \\
 \quad - 36a^2x^2 + 54a^3x \\
 \underline{\quad \quad \quad} \\
 * \quad - 54a^3x \\
 \quad - 54a^3x + 81a^4 \\
 \underline{\quad \quad \quad} \\
 * \quad *
 \end{array}$$

For another example of this kind, let it be required to divide $81x^4 - 256a^4$ by $3x + 4a$, and the work will stand thus:

$$\begin{array}{r}
 3x+4a) 81x^4 \quad * \quad * \quad * - 256a^4 \quad (27x^3 - 36ax^2 + 48a^2x - 64a^3 \\
 \underline{81x^4 + 108ax^3} \\
 * \quad - 108ax^3 \\
 \quad - 108ax^3 - 144a^2x^2 \\
 \underline{\quad \quad \quad} \\
 * \quad + 144a^2x^2 \\
 \quad + 144a^2x^2 + 192a^3x \\
 \underline{\quad \quad \quad} \\
 * \quad - 192a^3x \\
 \quad - 192a^3x - 256a^4 \\
 \underline{\quad \quad \quad} \\
 * \quad *
 \end{array}$$

As

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As division in decimal fractions may be often, by the help of cyphers, continued at pleasure, so likewise may division in Algebra by the help of stars: as if 1 was to be divided by $1+x$, the quotient would be $1-x+x^2-x^3+x^4-x^5$ &c. *ad infinitum*: but if 1 be divided by $1-x$, the quotient will be $1+x+x^2+x^3+x^4+x^5$ &c. *ad infinitum*. See the work:

$$\begin{array}{r}
 1+x) \quad 1 \quad * \quad * \quad * \quad * \quad * \quad (1-x+x^2-x^3+x^4-x^5 \text{ \&c.} \\
 \underline{1+x} \\
 * - x \\
 \underline{-x - x^2} \\
 * + x^2 \\
 \underline{+x^2 + x^3} \\
 * - x^3 \\
 \underline{-x^3 - x^4} \\
 * + x^4 \\
 \underline{+x^4 + x^5} \\
 * - x^5
 \end{array}$$

$$\begin{array}{r}
 1-x) \quad 1 \quad * \quad * \quad * \quad * \quad * \quad (1+x+x^2+x^3+x^4+x^5 \text{ \&c.} \\
 \underline{1-x} \\
 * + x \\
 \underline{+x - x^2} \\
 * + x^2 \\
 \underline{+x^2 - x^3} \\
 * + x^3 \\
 \underline{+x^3 - x^4} \\
 * + x^4 \\
 \underline{+x^4 - x^5} \\
 * + x^5
 \end{array}$$

Quotients arising from this sort of division are for the most part so regular, that a few terms may be sufficient to discover the whole process or continuation, as well as if the work was continued *ad infinitum*. As who that sees this work, and four or five of the first members of the quotient, can doubt of the rest?

K

For

For another example of this kind of division, let it be required to divide unity by the quantity $1-2x+xx$, and the work will be as follows:

$$\begin{array}{r}
 1-2x+xx \) \ 1 \quad * \quad * \quad * \quad * \quad (\ 1+2x+3xx+4x^3+5x^4 \ \&c. \\
 \underline{1-2x+xx} \\
 *+2x-xx \\
 \underline{+2x-4xx+2x^3} \\
 *+3xx-2x^3 \\
 \underline{+3xx-6x^3+3x^4} \\
 *+4x^3-3x^4 \\
 \underline{+4x^3-8x^4} \\
 *+5x^4 \\
 \underline{+5x^4}
 \end{array}$$

1st, By the four stars annexed to the dividend unity, is shewn, that this division, though productive of an infinite series, is not here intended to be continued beyond the fourth power of x .

2^{dly}, By the regularity of the quotient $1+2x+3xx+4x^3+5x^4$, none can doubt but that the next subsequent terms will be $+6x^5+7x^6+8x^7 \ \&c \ ad \ infinitum$.

3^{dly}, When quotients are thus regular, the very remainders will be so too: thus in the present case, the remainders are $2x-1x^2$, $3x^2-2x^3$, $4x^3-3x^4$, $5x^4-4x^5$; but of this last remainder, the latter part $-4x^5$ is dropped, as being out of the compass of the work; and so must all other terms be that are so, whether it be in multiplication, division, extraction of roots, or in any other operation whatever.

Let us now try the foregoing division by multiplication; that is, let the quotient $1+2x+3x^2+4x^3+5x^4$ be multiplied by the divisor $1-2x+xx$, and let us see whether the product will be $1*****$; for as the quotient is not true to above five places, it must not be expected that the product should be true to more. See the work, where the multiplication is carried on from left to right:

$$\begin{array}{r}
 1+2x+3x^2+4x^3+5x^4 \\
 1-2x+xx \\
 \hline
 1+2x+3x^2+4x^3+5x^4 \\
 -2x-4x^2-6x^3-8x^4 \\
 +x^2+2x^3+3x^4 \\
 \hline
 1 \quad * \quad * \quad * \quad *
 \end{array}$$

There

There is also another way of performing these multiplications, which I shall here put down; not so much for any extraordinary use it has in these cases, as that it may serve very well to explain Mr. *Oughtred's* rule for contracting multiplication, which see in his *Clavis*, chap. 4.

Here then the multiplier must be inverted, wherein the place of units must be put under that power of the multiplicand which is intended to be the last in the product; that is, in the present case, 1 must be placed under $5x^4$, thus:

$$\begin{array}{r} 1 + 2x + 3x^2 + 4x^3 + 5x^4 \\ x^2 - 2x + 1. \end{array}$$

Having thus put down the multiplicand and multiplier, every term of the multiplier must begin to multiply that term of the multiplicand which is next above it, and then must proceed to multiply all the other terms towards the left hand: thus 1 must multiply $5x^4 + 4x^3 + 3x^2 + 2x + 1$; but the next term of the multiplier, to wit, $-2x$ must multiply only $4x^3 + 3x^2 + 2x + 1$; and the last term of the multiplier, to wit, xx , must multiply only $3xx + 2x + 1$, as may be seen by the work, which follows:

$$\begin{array}{r} 1 + 2x + 3x^2 + 4x^3 + 5x^4 \\ x^2 - 2x + 1 \\ \hline 1 + 2x + 3x^2 + 4x^3 + 5x^4 \\ - 2x - 4x^2 - 6x^3 - 8x^4 \\ \hline x^2 + 2x^3 + 3x^4 \\ \hline 1 \quad * \quad * \quad * \quad * \end{array}$$

N. B. In the application of this example to *Oughtred's* rule, x must be made equal to $\frac{1}{10}$, xx to $\frac{1}{100}$, &c.

I could not well say less than I have done concerning division, to give the young Algebrist a tolerable notion of that operation; but to say more, and to enter into a detail of all the particular cases that may happen in literal division, would be to fill his head full of abstracted notions, which he is almost sure to forget, before ever he will come to apply them. It is for the same reason that I shall at this time pass by the invention of divisors, the doctrine of surds, the reduction of radicals, and some other things of the same nature, till we shall have a more immediate demand for them, and then I shall produce and explain them as we want them, without observing any other method.

Of Proportion in numbers.

15. The rule of proportion in Algebra is so very little different from the rule of proportion in common arithmetick, that one example of it will be sufficient. Let then the following question be put: *If a gives b, what will c give?* here the second and third terms multiplied together produce bc , and the quotient of this divided by the first term a , cannot otherwise be expressed than by the fraction $\frac{bc}{a}$: this is evident from

the notation of fractions explained in the 13th article. But as I have hitherto purposely avoided all consideration of proportion, choosing rather to appeal upon all occasions, to the common idea every one has or thinks he has of it, than to be more particular, it may not be improper, now we come to reason more closely upon things, to enter more distinctly into the particular nature of proportion, so far at least as it relates to numbers, and shew wherein it consists.

According to *Euclid*, four numbers are said to be proportionable, that is, the first number is said to have the same proportion to the second, that the third hath to the fourth; or the first is said to be to the second, as the third is to the fourth, when the first number is the same multiple, part or parts, of the second, that the third is of the fourth: but it will be asked perhaps; How can we know, what parts, part, or multiple, any one number is of another? To which I answer by a fraction, whose numerator is the former number, and denominator the latter: thus the fraction $\frac{2}{3}$ expressly shews, that the numerator 2 is two third parts of the denominator 3; for this is certain, that 1 is $\frac{1}{3}$ part of 3, and therefore 2 must be $\frac{2}{3}$ of it: for the same reason the fraction $\frac{12}{8}$ shews that the number 12 is $\frac{12}{8}$ or $\frac{3}{2}$ of the number 8; and lastly, the fraction $\frac{12}{4}$ shews, that the number 12 is $\frac{12}{4}$ of, or 3 times the number 4, and consequently, that 12 is a multiple of 4, as containing it just 3 times without any remainder: therefore to any one who understands fractions, *Euclid's* definition of proportion may be more distinctly expressed thus: *Four numbers are said to be proportionable, when a fraction whose numerator is the first number, and denominator the second, is equal to a fraction whose numerator is the third number and denominator the fourth.* Thus 2 is to 3 as 4 is to 6, because $\frac{2}{3}$ is equal to $\frac{4}{6}$; thus 12 is to 8 as 15 is to 10, because $\frac{12}{8}$ equals $\frac{15}{10}$, both being reducible to $\frac{3}{2}$; thus 2 is to 6 as 4 is to 12, because $\frac{2}{6}$ equals $\frac{4}{12}$, for each is equal to $\frac{1}{3}$; lastly, 6 is to 2 as 12 is to 4, because $\frac{6}{2} = \frac{12}{4} = 3$.

From this idea of proportionality may be demonstrated a very useful theorem in Algebra; which is, that *Whenever four numbers are proportionable,*

the

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the product of the extreme terms multiplied together will be equal to the product of the two middle terms so multiplied: for let a, b, c and d , be four proportionable numbers in their order; that is, let a be to b as c is to d ; I say then that ad the product of the extremes will be equal to bc the product of the two middle terms: for since a is to b as c is to d , it follows from what has already been laid down, that the fraction $\frac{a}{b}$ is equal to the fraction $\frac{c}{d}$; multiply both the terms of the fraction $\frac{a}{b}$ into d , and both those of the fraction $\frac{c}{d}$ into b , (which multiplications may be made without altering the values of the fractions,) and then you will have $\frac{ad}{bd} = \frac{bc}{bd}$; that is, the quotient of ad divided by bd , is equal to the quotient of bc divided by bd ; therefore ad must be equal to bc , that is, the product of the extremes must be equal to the product of the middle terms. Q. E. D.

The converse of this proposition is also true, to wit, that *Whenever we have an equation in numbers, wherein the product of two numbers on one side is found equal to the product of two numbers on the other, such an equation may be resolved into four proportionals, by making the two numbers on either side, the extremes, and those on the other side, the middle terms: thus if $ad = bc$; by making a and d the extremes, and b and c the middle terms, we shall have a to b as c to d : if this be denied, let a be to b as c is to e ; then we shall have $ae = bc$ by the last; but $ad = bc$ by the supposition; therefore $ae = ad$; therefore e equals d , and a is to b as c is to d . Q. E. D.*

COROLLARY.

Whence if a, b and c , be continual proportionals, that is, if a is to b as b is to c , we shall have $b^2 = ac$: and *e converso*, if $b^2 = ac$, then a, b and c will be continual proportionals.

The common properties of proportionality in numbers demonstrated.

16. From what has been delivered in the last article, may be demonstrated all or most of the common properties of proportionable numbers with a great deal of ease, some of the most useful whereof I shall here throw

throw together into one single article, for the reader to peruse, either at present, or hereafter, as he shall see occasion.

First then, from what has been said, may the rule of three, which consists in finding a fourth proportional, be most distinctly demonstrated: for let a , b and c be three numbers given, in order to find d , a fourth proportional; then since a is to b as c is to d , you will have ad the product of the extremes, equal to bc the product of the middle terms; di-

vide both sides of the equation by a , and you will have $d = \frac{bc}{a}$; which is as much as to say, that if three numbers be given, a fourth proportional may be obtained by multiplying the second and third numbers together, and dividing the product by the first.

In the rule of three inverse, let the numbers when disposed according to form be a , b and c ; then whosoever attentively considers the nature of that rule, will easily see, that the fourth number there sought for, is not to be a fourth proportional to the three numbers given as they are disposed in the order a , b , c , but as they stand in the order c , b , a , or c , a , b , and therefore in this case, the fourth number will be $\frac{ab}{c}$.

Secondly, if two proportions be equal to a third, they must be equal to one another, because if two fractions be equal to a third, they must be equal to one another: thus if a is to b as c is to d , and c is to d as e is to f , we shall have a to b as e to f .

Thirdly, if a is to b as c is to d ; then b will be to a as d to c , which is called inverse proportion: for if a is to b as c is to d , we shall have $ad = bc$; make b and c the extremes, and you will have b to a as d to c .

Fourthly, if a is to b as c is to d ; we shall have by permutation, a to c as b to d : for since a is to b as c is to d , and consequently $ad = bc$, make a and d the extremes, and c and b the middle terms, and you will have a to c as b to d .

Fifthly, if a is to b as c is to d , and any two multiplicators whatever be assumed, as e and f ; I say then, that ea is to fb , as ec to fd : for since a is to b as c is to d , and so $ad = bc$; multiply both sides of the equation by the product ef , and you will have $ad \times ef = bc \times ef$; but $ad \times ef = ea \times fd$, and $bc \times ef = fb \times ec$; therefore $ea \times fd = fb \times ec$; make ea and fd extremes, and the proportion will stand thus; ea is to fb as ec to fd . In like manner, *mutatis mutandis*, it may be demonstrated, that if a is to b as c is to d , then $\frac{a}{e}$ will be to $\frac{b}{f}$ as $\frac{c}{e}$ is to $\frac{d}{f}$.

Sixthly, if a is to b as c is to d ; then a^2 is to b^2 as c^2 is to d^2 : for since a is to b as c is to d , and so $ad = bc$; square both sides of the equation,

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equation, and you will have $a^2 d^2 = b^2 c^2$; make a^2 and d^2 extremes, and you will have a^2 to b^2 as c^2 to d^2 . And by taking these steps backwards, it will also appear, that if a^2 is to b^2 as c^2 is to d^2 ; a is to b as c is to d , and \sqrt{a} is to \sqrt{b} as \sqrt{c} is to \sqrt{d} .

Seventhly, if a is to b as c is to d ; then by composition (as it is called) $\overline{a+b}$ is to b as $\overline{c+d}$ is to d ; or $\overline{a+b}$ is to a , as $\overline{c+d}$ is to c : for since a is to b as c is to d , and consequently $ad = bc$; add bd to both sides of the equation, and you will have $ad + bd = bc + bd$; but $ad + bd$ is the product of $\overline{a+b}$ multiplied into d , as is easily seen; and $bc + bd$ is the product of b multiplied into $\overline{c+d}$; therefore $\overline{a+b} \times d = b \times \overline{c+d}$; make $\overline{a+b}$ and d extremes, and you will have $\overline{a+b}$ to b as $\overline{c+d}$ to d . Again, since $bc = ad$, add ac to both sides, and you will have $ac + bc = ac + ad$, that is, $\overline{a+b} \times c = a \times \overline{c+d}$; make $\overline{a+b}$ and c extremes, and you will have $\overline{a+b}$ to a as $\overline{c+d}$ to c .

Eighthly, if a is to b as c is to d ; then by division $\overline{a-b}$ is to b as $\overline{c-d}$ is to d ; or $\overline{a-b}$ is to a as $\overline{c-d}$ to c . This proposition is demonstrated by subtraction, just in the same manner as the last was by addition.

Ninthly, if to or from two numbers in any given proportion, be added or subtracted other two numbers in the same proportion, the sums or remainders will still be in the same proportion with the numbers first proposed: thus if the numbers c and d be in the same proportion with the numbers a and b , that is, if as a is to b so is c to d , and if to or from the former two numbers, be added or subtracted the latter, we shall have not only $\overline{a+c}$ to $\overline{b+d}$ as a to b , but also $\overline{a-c}$ to $\overline{b-d}$ as a to b : for since by the supposition, a is to b as c is to d ; it follows by permutation, that a is to c as b is to d ; and by composition, that $\overline{a+c}$ is to a as $\overline{b+d}$ to b ; and again by permutation, that $\overline{a+c}$ is to $\overline{b+d}$ as a is to b : in like manner by permutation and division we shall have $\overline{a-c}$ to $\overline{b-d}$ as a to b .

Tenthly, if there be three numbers a , b and c , and other three numbers d , e and f proportionable to them, and in the same order, that is, if as a is to b so d is to e , and as b is to c so e is to f ; I say then, that *ex aequo*, the extremes will be in the same proportion, (*viz.*) that a will be to c as d is to f : for since by the supposition, a is to b as d is to e ; by permutation we shall have a to d as b to e ; and for the same reason, since b is to c as e is to f ; we shall have b to e as c to f : since then a is

to d as b to e , and b to e as c to f ; it follows from the second proposition, that a is to d as c to f ; and by permutation, that a is to c as d to f .

Eleventhly, if there be three numbers, a , b and c , and three other numbers d , e and f proportionable to them, but in a contrary order, so that a is to b as e to f , and b to c as d to e ; I say, that the extremes will still be proportionable, to wit, that a will be to c as d to f : for since a is to b as e to f , we have $af = be$; moreover since b is to c as d to e , we have $cd = be$; therefore $af = cd$; make a and f extremes, and you will have a to c as d to f .

N. B. If there be two serieses of numbers as $a, b, c, \&c; d, e, f, \&c;$; each series consisting of the same number of terms; and if all the proportions between contiguous terms in one series, be respectively equal to all those in the other, that is, each to each, as they stand in order; as if a be to b as d to e , and b to c as e to f , $\&c;$; then the extreme terms of one series will be proportionable to the extreme terms of the other: for the demonstration of the tenth proposition may be extended to as many terms as we please; and this proportionality of the extremes, is said to follow *ex æquo ordinate*, or barely *ex æquo*, that is, from a respective equality of all the proportions in one series to their correspondents in the other, in an orderly manner. But if every proportion in one series, has an equal proportion to answer it in the other, but not in a correspondent part of the series; as if a be to b as e to f , and b to c as d to e , $\&c;$; then though the extremes will still be proportionable, as will be evident by continuing the demonstration of this eleventh proposition; yet now the proportionality of the extremes is said to follow *ex æquo perturbate*, that is, from an equality of all the proportions in one series to all those in the other, but in a disorderly manner.

Twelfthly, if a is to b as c is to d ; we shall have $\overline{a+b}$ to $\overline{a-b}$ as $\overline{c+d}$ is to $\overline{c-d}$: for since a is to b as c is to d , we shall have by composition, $\overline{a+b}$ to a as $\overline{c+d}$ is to c ; we shall have also by division, $\overline{a-b}$ to a as $\overline{c-d}$ to c ; and by inversion, a to $\overline{a-b}$ as c to $\overline{c-d}$: since then we have $\overline{a+b}$ to a as $\overline{c+d}$ to c ; and a to $\overline{a-b}$ as c to $\overline{c-d}$, that is, since we have three numbers, $\overline{a+b}$, a , and $\overline{a-b}$, and other three numbers proportionable to them in the same order, to wit, $\overline{c+d}$, c , and $\overline{c-d}$; it follows *ex æquo*, that the extremes will be proportionable, that is, that $\overline{a+b}$ will be to $\overline{a-b}$ as $\overline{c+d}$ is to $\overline{c-d}$.

Thirteenthly, if there be a series of numbers, k, l, m, n , whereof k is to l as a to b , and l to m as c to d , and m to n as e to f ; I say then

that

Art. 16, 17, 18. *Extraction of the square roots of algebraic quantities.* 81
 that k the first term, will be to n the last, as ace the product of all the other antecedents to bdf the product of all the other consequents: for k is to l as a to b , by the supposition; and we shall find that a is to b as ace to bce by multiplying extremes and means; therefore k is to l as ace to bce ; and for a like reason l is to m as bce to bde , and m is to n as bde to bdf ; therefore *ex æquo*, k is to n as ace to bdf .

Of the extraction of the square roots of simple algebraic quantities.

17. The extraction of the square root of simple algebraic quantities is so very easy, that it needs not to be insisted on. Thus the square root of aa is $+$ or $-a$, the square root of $9aa$ is $+$ or $-3a$, and that of $4aabb$ is $+$ or $-2ab$: this is plain from the definition of the square root; for the square root of any quantity, suppose of $4aabb$, is that, which being multiplied into itself, will produce $4aabb$: now $-2ab$ multiplied into itself will produce $4aabb$, as well as $+2ab$, and therefore one quantity is as much its square root as the other.

When the square root of a quantity cannot be extracted, it is usual to signify it by this mark $\sqrt{}$: thus $\sqrt{2aa}$ signifies the square root of $2aa$; thus $\sqrt{aa-4b}$ signifies the square root of the whole quantity $aa-4b$; thus $\frac{\sqrt{aa-4b}}{2a}$ signifies a fraction whose numerator is the square

root of the whole quantity $aa-4b$, and whose denominator is $2a$; thus $\frac{\sqrt{4ab-a^3}}{12a}$ signifies the square root of the whole fraction $\frac{4ab-a^3}{12a}$; that is, the square root of both the numerator and denominator.

When the square root of a quantity cannot be extracted, the quantity may sometimes however be resolved into two factors, whereof the one is a square, and the other is not; and whenever this is possible, the root of the square may be extracted, and the radical sign may be prefixed to the other factor: thus $12aa$ equals $4aa \times 3$; therefore $\sqrt{12aa} = 2a\sqrt{3}$.

Of the extraction of the square roots of compound algebraic quantities.

18. The extraction of the square root of compound algebraic quantities is so very like that of whole numbers in common Arithmetick, especially in the case of serieses where it is chiefly required, that one would be apt to imagine, a bare inspection of the work would be sufficient to shew the process to any one tolerably well skilled in the latter; but if that be not thought sufficient, take the following directions, with an example.

L

Let

Let it be required to extract the square root of this quantity $x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16$: here, pointing every other place from the place of units, that is, from 16, I demand the square root of the member belonging to the first point to the left hand, that is, of x^6 , and the answer is x^3 , which I put down in the quotient; then I square x^3 , and subtract the product x^6 , from x^6 , the member belonging to the first point, and there remains nothing: then I bring down the two next members belonging to the next point, to wit, $4x^5 + 10x^4$ for a resolvend, and divide it by $2x^3$, the double of the root in the quotient, that is, I divide $4x^5$ the first member of the resolvend, by $2x^3$, and the quotient is $2x^2$, which I put down in the quotient, as likewise after the divisor $2x^3$; then I multiply the quantity $2x^3 + 2x^2$ by it's last member $2x^2$, and the product $4x^5 + 4x^4$ I subtract from the resolvend $4x^5 + 10x^4$, and there remains $6x^4$; to this remainder I bring down the next two places $20x^3 + 25x^2$, and so have a new resolvend, $6x^4 + 20x^3 + 25x^2$; this resolvend I divide by the double of the root already found, that is, by $2x^3 + 4x^2$, dividing the first member of the resolvend by the first member of this double root, and the quotient is $+3x$, which I put down in the quotient, and also after the divisor $2x^3 + 4x^2$; then multiplying this last quantity $2x^3 + 4x^2 + 3x$ by it's last member $3x$, the product is $6x^4 + 12x^3 + 9x^2$, which being subtracted from the last resolvend, leaves $8x^3 + 16x^2$; to this remainder I bring down the two last places $24x + 16$, and so have a new resolvend $8x^3 + 16x^2 + 24x + 16$; this I divide by the double of the root already found, that is, by $2x^3 + 4x^2 + 6x$, and the quotient is $+4$; this being put down in the quotient, and also after the divisor $2x^3 + 4x^2 + 6x$ gives $2x^3 + 4x^2 + 6x + 4$; which being multiplied by it's last member 4, gives $8x^3 + 16x^2 + 24x + 16$, which being subtracted from the last resolvend, leaves no remainder; therefore the whole root is $x^3 + 2x^2 + 3x + 4$. See the work:

$$\begin{array}{r}
 x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16 \quad (x^3 + 2x^2 + 3x + 4) \\
 \underline{x^6} \\
 4x^5 + 10x^4 \\
 \underline{4x^5 + 4x^4} \\
 6x^4 + 20x^3 + 25x^2 \\
 \underline{6x^4 + 12x^3 + 9x^2} \\
 8x^3 + 16x^2 + 24x + 16 \\
 \underline{8x^3 + 16x^2 + 24x + 16} \\
 0
 \end{array}$$

Other.

Otherwise thus:

$$\begin{array}{r}
 x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16 \quad (x^3 + 2x^2 + 3x + 4) \\
 \hline
 2x^3 + 2x^2 \quad * \\
 \quad + 2x^2 \quad + 4x^5 \quad + 4x^4 \\
 \hline
 2x^3 + 4x^2 + 3x \quad * \quad + 6x^4 \\
 \quad + 3x \quad \quad \quad 6x^4 + 12x^3 + 9x^2 \\
 \hline
 2x^3 + 4x^2 + 6x + 4 \quad * \quad + 8x^3 + 16x^2 \\
 \quad + 4 \quad \quad \quad 8x^3 + 16x^2 + 24x + 16 \\
 \hline
 \quad \quad \quad * \quad * \quad * \quad *
 \end{array}$$

Since the last member $+16$ of the quantity given, was a square, I might have begun the extraction at that end, and the root would have come out the same as before, but inverted.

N. B. By a square number I mean a number that admits of a square root; therefore -16 is no square number, since there is no root either affirmative or negative, which multiplied into itself will produce -16 .

For another example, let it be proposed to extract the square root of the following quantity, or at least to reduce the root to the simplest expression, $12x^3 - 72x^2 + 108x$: this quantity, because neither of the extremes are squares, must be reduced to a more convenient form thus; if $12x^3$ be divided by $12x$, the quotient xx will be a square number; therefore I divide the whole by $12x$, and the quotient is $xx - 6x + 9$; whence I conclude *e converso*, that if $xx - 6x + 9$ be multiplied into $12x$, the product will be the quantity proposed, to wit, $12x^3 - 72x^2 + 108x$; therefore by the 8th article, $\sqrt{12x^3 - 72x^2 + 108x}$ equals $\sqrt{xx - 6x + 9}$ into $\sqrt{12x}$: but $\sqrt{x^2 - 6x + 9}$ being extracted as above, is $x - 3$, or $3 - x$; and because $12x = 4 \times 3x$, we have $\sqrt{12x} = 2 \times \sqrt{3x}$; therefore, $\sqrt{12x^3 - 72x^2 + 108x} = x - 3$ or $3 - x$ into $2 \times \sqrt{3x} = 2x - 6$ or $6 - 2x$ into $\sqrt{3x}$. If the square root of the factor $xx - 6x + 9$ could not otherwise have been extracted, it might however have been obtained by an approximation, as is that of a binomial in the following article.

To extract the square root of a binomial by an infinite series.

19. By a binomial in this place I mean any quantity consisting of two simple quantities connected together by the sign $+$ or $-$, as $a+b$, $1-x$, &c: now though the square root of such a quantity cannot be expressed by any finite number of terms, yet by an infinite series it may, as in the following cases.

Case the 1st. The square root of the binomial $1+x$, supposing x to be less than one, is found by the following operation to be $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\&c$; a series, whose regularity is not immediately perceived, but I shall make it appear in the course of this work, that these serieses are as regular as any others whatever, and as easily computed.

$$\begin{array}{r}
 1+x \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad (1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\&c. \\
 \hline
 1 \\
 2+\frac{x}{2} \quad * \quad +x \\
 \hline
 +\frac{x}{2} \quad x+\frac{x^2}{4} \\
 \hline
 2+x-\frac{x^2}{8} \quad * \quad -\frac{x^2}{4} \\
 \hline
 -\frac{x^2}{8} \quad -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
 \hline
 2+x-\frac{x^2}{4}+\frac{x^3}{16} \quad * \quad +\frac{x^3}{8}-\frac{x^4}{64} \\
 \hline
 +\frac{x^3}{16} \quad \frac{x^3}{8}+\frac{x^4}{16} \\
 \hline
 2+x-\frac{x^2}{4}+\frac{x^3}{8}-\frac{5x^4}{128} \quad * \quad -\frac{5x^4}{64} \\
 \hline
 -\frac{5x^4}{128} \quad -\frac{5x^4}{64} \\
 \hline
 *
 \end{array}$$

Since

Art. 19. OF ALGEBRAIC QUANTITIES. 85

Since in all these serieses, the powers of x ascend or descend regularly, these roots may be extracted by the help of the coefficients only, and the powers may be supplied afterwards; but then due care must be taken to keep the terms distinct, so as to place all of the same kind orderly one under another, that no confusion may arise in the operation.

Let us now try the root thus found, and see what it will produce when multiplied into itself: but first I must advertise the reader, that as this root is not extended beyond the fourth power of x , so neither ought it to be expected that the square of this root when multiplied into itself, should be true to more places than the root; therefore in this multiplication, all the powers of x beyond the fourth ought to be excluded out of the product as useless terms. Now in order to square the quantity

$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$, it must be multiplied,

First into 1, and the product will be	$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \&c.$
then into $\frac{1}{2}x$, and the product will be	$+\frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{16}x^3 + \frac{1}{32}x^4 \&c.$
then into $-\frac{1}{8}x^2$, and the product will be	$-\frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{1}{64}x^4 \&c.$
then into $\frac{1}{16}x^3$, and the product will be	$+\frac{1}{16}x^3 + \frac{1}{32}x^4 \&c.$
then into $-\frac{5}{128}x^4$, and	$-\frac{5}{128}x^4 \&c.$
the product will be	

add these together, and the sum will be $1 + x \quad * \quad * \quad *$
as it ought.

As in squaring this root, no notice was taken of any powers of x beyond the fourth, so likewise in extracting the square root, the work must not be suffered to run on beyond the power where the series is designed to terminate, that is, all other powers must be excluded out of the work, as they are out of the root.

Case 2d. Let now the square root of the binomial $1-z$ be required: where it is plain that z must be less than 1, otherwise $1-z$ would be a negative quantity, and could have no square root: put then $-z$ instead of x in the foregoing case, and you will have $xx = +z^2$, $x^3 = -z^3$, $x^4 = +z^4 \&c$; $\frac{1}{2}x = -\frac{1}{2}z$, $-\frac{1}{8}x^2 = -\frac{1}{8}z^2$, $+\frac{1}{16}x^3 = -\frac{1}{16}z^3$, $-\frac{5}{128}x^4 = -\frac{5}{128}z^4 \&c$: therefore $\sqrt{1-z} = 1 - \frac{1}{2}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \frac{5}{128}z^4 - \&c.$

Case 3d. Let the square root of the binomial $z+1$ be required, supposing z to be greater than 1: here, though 1 be a square number, yet as this root is to be expressed by an infinite series perpetually converging nearer and nearer to it; the series must take it's rise from the greater part of the binomial, that is, from z : divide then the quantity $z+1$ by z ,
and

and the quotient will be $1 + \frac{1}{z}$, where $\frac{1}{z}$ will be less than 1, because z is greater: since then $\frac{z+1}{z}$ equals $1 + \frac{1}{z}$, it follows *e converso*, that $\sqrt{1 + \frac{1}{z}} \times z = z + 1$; therefore $\sqrt{1 + \frac{1}{z}} \times \sqrt{z} = \sqrt{z+1}$: put $\frac{1}{z}$ instead of x in the first case, and you will have $\sqrt{1 + \frac{1}{z}} = 1 + \frac{1}{2z} - \frac{1}{8z^2} + \frac{1}{16z^3} - \frac{5}{128z^4} \&c$: call the sum of this series continued *ad infinitum* s , and you will have $\sqrt{z+1} = s\sqrt{z}$, that is, $s \times \sqrt{z}$.

Case 4th. Let it be required to express by an infinite series, the square root of the binomial $a+b$, supposing a to be the greater part of the binomial: here $\frac{a+b}{a} = 1 + \frac{b}{a}$; therefore $\sqrt{1 + \frac{b}{a}} \times a = a + b$; but $\sqrt{1 + \frac{b}{a}}$ by the first case is $1 + \frac{b}{2a} - \frac{b^2}{8a^2} + \frac{b^3}{16a^3} - \frac{5b^4}{128a^4} \&c$: call the sum of this series continued *ad infinitum* s , and you will have $\sqrt{a+b} = s\sqrt{a}$.

Preparations for demonstrating the rule for finding the greatest common measure of two numbers given; that is, their greatest common divisor.

20. Coming now to give some examples of the several reductions and operations of algebraic fractions; I shall take this occasion before I begin, to give some account of the rule for finding the greatest common measure to any two numbers proposed; a rule very useful in vulgar fractions, and sometimes in Algebra, when both the numerator and denominator of a fraction are compound quantities.

This should have been done sooner, but that I made some doubt whether such a demonstration of this rule could be given, as might be suited to the capacity of a young beginner; nay I am not absolutely certain whether even still he will be able to comprehend it: but if he finds it too difficult for him, the best advice I can give him is, to pass it by till he finds his imagination more confirmed, his attention more fixed, and his reason improved by habit and exercise.

For

Art. 20, 21. COMMON MEASURE DEMONSTRATED.

87

For the better understanding of this rule and the demonstration belonging to it, I shall lay down the following definitions and axioms.

Def. 1. *One number is said to measure another, when it is exactly contained in that other without any overplus or remainder.* Thus 3 may be said to measure 12, as being exactly four times contained in it.

2. *A number is said to be a common measure to two others, when it measures them both.* Thus 3 is a common measure both to 12 and 21.

Axiom 1st.

If one quantity measures another, and that other measures a third, then will the first measure the third. As if 3 measures 12, and 12 measures 24, 3 will measure 24.

Axiom 2d.

If a number be a common measure to two others, it will measure both their sum and their difference. Thus 3, which is a common measure to 12 and 21, will measure both their sum 33, and their difference 9; for if 3 be four times contained in 12, and seven times in 21, it ought to be eleven times contained in their sum, because $4+7=11$; and it ought to be three times contained in their difference, because $7-4=3$.

Axiom 3d.

If one number be divided by another, and there be any remainder, take away the remainder from the dividend, and the divisor will measure the rest. Thus if 14 be divided by 3, and there remains 2, take away 2 from 14, and 3 will measure the rest, which is 12. In general thus: *If a be divided by b, and there remains c, then will b measure $a-c$.*

The rule explained and demonstrated.

21. These things being laid down, I come now to the rule for finding the greatest common measure to any two numbers proposed, which is as follows: *Let a and b be the two numbers whose greatest common measure is wanted, to wit, a the greater, and b the less: let a be divided by b, and without any regard had to the quotient, let c be the remainder; then divide b by c, and let the remainder be d; then divide c by d, and let the remainder be e; lastly divide d by e, and let there be no remainder at all; I say then, that this last divisor e, which had no remainder, will not only be a common measure to a and b, but will be the greatest common measure they will admit of.*

But

But first I shall prove, that e is a common measure to a and b , which is thus done :

1st. e measures d *ex hypothesi*; and d measures $\overline{c-e}$, by the third axiom, because when c was divided by d , the remainder was e ; therefore e measures $\overline{c-e}$, by the first axiom; but e measures itself; therefore e measures both e and $\overline{c-e}$; but the sum of e and $\overline{c-e}$ is c ; therefore e measures c by the second axiom.

2^{dly}. It has now been proved that e measures c ; but c measures $\overline{b-d}$; therefore e measures $\overline{b-d}$; but e measures d *ex hypothesi*; therefore e measures both d and $\overline{b-d}$; but the sum of d and $\overline{b-d}$ is b ; therefore e measures b .

3^{dly}. It has been proved that e measures b ; and b measures $\overline{a-c}$; therefore e measures $\overline{a-c}$; but it was proved before, that e measures c ; therefore e measures both c and $\overline{a-c}$; but the sum of c and $\overline{a-c}$ is a ; therefore e measures a . Thus then we have proved the first part of our assertion, which was, that e measures both a and b .

We shall in the next place, by tracing back the steps of the former demonstration, prove, that e is the greatest quantity that will measure them both: for, if possible, let us suppose f to be greater than e , and yet to be a common measure to a and b , and let us see what will be the consequence of this supposition.

1st. f measures b , and b measures $\overline{a-c}$; therefore f measures $\overline{a-c}$; but f measures a by the supposition; therefore f measures both a and $\overline{a-c}$; but the difference between a and $\overline{a-c}$ is c ; therefore f measures c , by the second axiom.

2^{dly}. f measures c , as above, and c measures $\overline{b-d}$; therefore f measures $\overline{b-d}$; but f measures b by the supposition; therefore f measures both b and $\overline{b-d}$; but the difference between b and $\overline{b-d}$ is d ; therefore f measures d .

3^{dly}. Again, f measures d as above, and d measures $\overline{c-e}$; therefore f measures $\overline{c-e}$; but f measures c , as was proved before; therefore f measures both c and $\overline{c-e}$; but the difference between c and $\overline{c-e}$ is e ; therefore f measures e , that is, a greater quantity measures a less, which is absurd; therefore the supposition that a and b would admit of a greater common measure than e , from whence this absurdity flowed, was false; for truth never leads to absurdities: therefore e is the greatest common measure the two numbers a and b will admit of. Q. E. D.

N. B.

Art. 21, 22. *Examples of fractions in algebraic quantities.* 89

N. B. This demonstration is *Euclid's* a little changed, and may serve as a specimen of the subtilty of the Ancients.

For an example to the foregoing rule, see the introduction, art. 7th.

C O R O L L A R I E S.

1st. If in finding the greatest common measure we can have no divisor without any remainder till we come to unity, we ought then to conclude, that the numbers proposed are *primi inter se*, that is, are such as will admit of no common measure but unity: for *As every number is called a prime number, that will admit of no divisor but itself and unity; so two numbers are said to be prime to each other, that will admit of no common divisor but unity.*

2d. *Whatever number will measure two others, it will also measure their greatest common measure:* thus upon the supposition that *f* would measure both *a* and *b*, it was proved to measure *e*, their greatest common measure.

The several rules of fractions exemplified in Algebraic quantities.

22. Fractions in Algebra are treated just in the same manner as in common arithmetic, only using algebraical instead of numeral operations; as will plainly appear from the following examples.

Examples of the reduction of fractions from higher to lower terms, according to introduction art. 7th.

The fraction $\frac{4ab}{6bc}$, dividing both the numerator and denominator by the same quantity $2b$, will be reduced to the fraction $\frac{2a}{3c}$, a fraction of the same value with the former, but expressed in more simple terms: whence we may infer, that whenever a common letter or factor is to be found in every member both of the numerator and denominator, it may be cancelled every where, without affecting the value of the fraction: thus the fraction $\frac{ac+bc}{cd+ce}$, expunging c , becomes $\frac{a+b}{d+e}$, a fraction of the same value. But if there be any one member, wherein the factor is not concerned, it must not be expunged at all: thus the fraction $\frac{ac+bc}{cd+e}$ cannot be reduced, because the factor c is not to be found in e .

Note, That cancelling here, is not subtracting, but dividing: thus to cancel the letter b in the quantity ab , so as to reduce it to a ; is not to

M

subtract

subtract b from ab , but to divide ab by b , in which case the quotient will be a .

Examples of fractions reduced to the same denomination, according to introduction art. 8th.

1st. The fractions $\frac{a}{2}$, $\frac{b}{3}$ and $\frac{c}{4}$, when reduced to the same denomination, will stand thus; $\frac{12a}{24}$, $\frac{8b}{24}$ and $\frac{6c}{24}$. 2d. The fractions $\frac{a}{b}$ and $\frac{c}{d}$ so reduced, will stand thus; $\frac{ad}{bd}$ and $\frac{bc}{bd}$. 3d. The fractions $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{u}$ and $\frac{x}{y}$, after reduction, will stand thus; $\frac{psuy}{qsuy}$, $\frac{qr uy}{qsuy}$, $\frac{qsty}{qsuy}$ and $\frac{qsux}{qsuy}$. And here I cannot but observe, that now the rule for this reduction demonstrates itself: for in this example it is impossible not to see, that all these fractions, notwithstanding this reduction, still retain their former values: thus the first fraction $\frac{psuy}{qsuy}$, by cancelling common factors, is reduced to $\frac{p}{q}$, it's former value; and the same may be observed of all the rest: and this example amounts to a demonstration, because it is comprehended in general terms. But to go on: 4th. The fractions $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ being reduced to the same denomination, become $\frac{bc}{abc}$, $\frac{ac}{abc}$ and $\frac{ab}{abc}$. 5th. And lastly, $\frac{1}{a+b}$ and $\frac{1}{a-b}$ when thus reduced, become $\frac{a-b}{aa-bb}$ and $\frac{a+b}{aa-bb}$: for 1 the numerator of the first fraction multiplied into $a-b$, the denominator of the second, makes $a-b$; and 1 the numerator of the second fraction multiplied into $a+b$, the denominator of the first, makes $a+b$; and the product of the two denominators $a+b$ and $a-b$ multiplied together is $aa-bb$, as in the 4th example of the 9th article.

Examples of addition in fractions, according to introduction art. 9th.

1st. These fractions $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{-c}{2}$, when added together, make $\frac{a+b-c}{2}$.

2d.

2d. The fraction $\frac{a+b}{2}$ added to the fraction $\frac{a-b}{2}$ makes $\frac{2a}{2}$ or a .

3d. The fractions $\frac{a}{2}$, $\frac{-b}{3}$, and $\frac{+c}{4}$ when added together, make

$$\frac{12a-8b+6c}{24}.$$

4th. The fraction $\frac{a}{b}$ added to the fraction $\frac{c}{d}$ makes $\frac{ad+bc}{bd}$.

5th. a added to $\frac{b}{c}$, that is, $\frac{a}{1}$ added to $\frac{b}{c}$ makes $\frac{ac+b}{c}$,

6th. $\frac{1}{a}$ added to $-\frac{1}{b}$ makes $\frac{b-a}{ab}$.

7th. The fractions $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{u}$ and $\frac{x}{y}$, when added together, make

$$\frac{psuy+qrux+qsty+qsux}{qsuy}.$$

8th. $\frac{a}{b}$ added to $\frac{1}{c}$ gives $\frac{ac+b}{bc}$.

9th. $\frac{1}{a+b}$ added to $\frac{1}{a-b}$ gives $\frac{2a}{aa-bb}$: See the 5th example of fractions reduced to the same denomination.

Examples of subtraction in fractions, according to introduction art. 10th.

Note first, If the signs of both the numerator and denominator of any fraction be changed, which is no more than multiplying both terms into -1 , the value of the fraction will still remain.

Secondly, The denominator of a fraction is always supposed to be affirmative; and therefore if at any time it happens to be otherwise, it must be made affirmative by changing the signs of both terms.

Thirdly, $+\frac{a}{b}$ and $-\frac{a}{b}$ are the same in effect as $\frac{+a}{b}$ and $\frac{-a}{b}$, as is evident from the nature of division: and sometimes, this latter way of notation is more convenient than the former.

Fourthly, Therefore the sign of the numerator is the sign of the whole fraction; and to change the sign of the former, is the same in effect, as to change the sign of the latter.

Fifthly, Whenever one algebraic fraction is to be subtracted from another, the safest way will be to change the sign of the numerator of the fraction to be subtracted, and to place it after the other, and then to reduce them at last into one fraction: for if the subtraction be deferred till after the reduction is over, one may make a mistake, and subtract the

wrong quantity. Thus 1st. $\frac{4b}{5}$ subtracted from $\frac{2a}{3}$ gives $\frac{2a}{3} - \frac{4b}{5} = \frac{10a - 12b}{15}$.

2d. $\frac{r}{s}$ subtracted from $\frac{p}{q}$ gives $\frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}$.

3d. $\frac{b}{c}$ subtracted from a , gives $a - \frac{b}{c} = \frac{ac - b}{c}$.

4th. $\frac{1}{a+b}$ subtracted from $\frac{1}{a-b}$, gives $\frac{1}{a-b} - \frac{1}{a+b} = \frac{2b}{a^2 - b^2}$.

Examples of multiplication in fractions.

The multiplication of fractions is performed, by multiplying the numerator and denominator of the multiplicand, into the numerator and denominator of the multiplier respectively.

Thus 1st. $\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}$.

2d. $\frac{3p}{4q} \times \frac{5q}{6r} = \frac{15pq}{24qr} = \frac{5p}{8r}$.

3d. $\frac{a}{b} \times c$ or $\frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}$.

4th. $\frac{a}{b} \times b = \frac{ab}{b} = a$.

5th. $\frac{3a}{4b} \times 20b = \frac{60ab}{4b} = 15a$.

6th. $\frac{4a}{5} \times \frac{7}{8a} = \frac{28a}{40a} = \frac{7}{10}$.

7th. $\frac{3a}{4b} \times \frac{3a}{4b} = \frac{9aa}{16bb}$.

8th.

$$8\text{th. } \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$$

$$9\text{th. } a + \frac{b}{c} \times d, \text{ or } \frac{ac+b}{c} \times \frac{d}{1} = \frac{acd+bd}{c}.$$

$$10\text{th. } d + \frac{e}{f} \times \frac{g}{b}, \text{ or } \frac{df+e}{f} \times \frac{g}{b} = \frac{dfg+eg}{fb}.$$

$$11\text{th. } a + \frac{b}{c} \times d + \frac{e}{f}, \text{ or } \frac{ac+b}{c} \times \frac{df+e}{f} = \frac{acdf+ace+ddf+be}{cf}.$$

This multiplication might also be performed thus :

$$\begin{array}{r} d + \frac{e}{f} \\ a + \frac{b}{c} \\ \hline ad + \frac{ae}{f} + \frac{bd}{c} + \frac{be}{cf} \end{array}$$

$$12\text{th. } a + \frac{b}{c} \times a + \frac{b}{c} = \frac{aacc+2abc+bb}{cc}$$

$$\text{Or } aa + \frac{2ab}{c} + \frac{bb}{cc}. \text{ See the work :}$$

$$\begin{array}{r} a + \frac{b}{c} \\ a + \frac{b}{c} \\ \hline aa + \frac{ab}{c} + \frac{bb}{cc} \\ + \frac{ab}{c} \\ \hline aa + \frac{2ab}{c} + \frac{bb}{cc} \end{array}$$

Examples

Examples of division in fractions.

Division in fractions is performed by multiplying the direct terms of the dividend into the inverted terms of the divisor: thus,

$$1^{\text{st}}. \frac{r}{s}) \frac{p}{q} \left(\frac{ps}{qr}. \quad 2^{\text{d}}. \frac{b}{c}) \frac{1}{a} \left(\frac{c}{ab}.$$

$$3^{\text{d}}. \frac{1}{c}) \frac{a}{b} \left(\frac{ac}{b}. \quad 4^{\text{th}}. c) \frac{a}{b} \left(\frac{a}{bc}.$$

$$5^{\text{th}}. \frac{1}{c}) \frac{ab}{c} \left(\frac{abc}{c}, \text{ or } ab. \quad 6^{\text{th}}. d + \frac{e}{f}) a + \frac{b}{c} \left(\frac{acf + bf}{cdf + ce}.$$

$$7^{\text{th}}. \sqrt{b}) \sqrt{a} \left(\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}: \text{ for if we make } x = \frac{\sqrt{a}}{\sqrt{b}} \text{ we shall have } xx = \frac{a}{b}, \text{ and } x = \sqrt{\frac{a}{b}}.$$

I shall only give one example more, and that shall be of the rule of proportion, as follows: If $\frac{a}{b}$ gives $\frac{c}{d}$ what will $\frac{e}{f}$ give? Answer, $\frac{bce}{adf}$: for $\frac{c}{d}$ the second number multiplied into $\frac{e}{f}$ the third, produces $\frac{ce}{df}$; and this divided by the first $\frac{a}{b}$ quotes $\frac{bce}{adf}$.

Of equations in Algebra; and particularly of simple equations, together with the manner of resolving them.

23. An equation in Algebra is a proposition wherein one quantity is declared equal to another, or where one expression of any quantity is declared equal to another expression of the same quantity: as when we say $\frac{2}{4} = \frac{1}{2}$; where $\frac{2}{4}$ is said to possess one side of the equation, and $\frac{1}{2}$ the other.

An affected quadratic equation is an equation consisting of three different sorts of quantities; one wherein the square of the unknown quantity is concerned, another wherein the unknown quantity is simply concerned, and a third wherein it is not concerned at all: as if $xx - 2x = 3$; supposing x to be an unknown quantity.

If either the term wherein the simple power of x is concerned, as $-2x$, or that which is called the absolute term, to wit, 3, be wanting, the equation

Art. 23. RESOLUTION OF SIMPLE EQUATIONS.

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tion is still a quadratic equation, though incomplete. Some indeed there are, who rank this latter sort of equations under the denomination of simple equations; and so shall we, upon account of their easy resolution; though properly speaking, a simple equation is that wherein some simple power of the unknown quantity is concerned, all others being excluded: as if $3x=6$; $2x+3=4x-5$, &c.

The use of these equations is for representing more conveniently and more distinctly the conditions of problems, when translated out of common language into that of Algebra. As for example; let it be proposed to find a number with the following property, to wit, that $\frac{2}{3}$ of it with 4 over, may amount to the same as $\frac{7}{12}$ of it with 9 over: here, putting x for the unknown quantity, the condition of this problem, when translated out of common language into that of Algebra, will be represented by the following equation, to wit,

$$\frac{2x}{3} + 4 = \frac{7x}{12} + 9 : \text{for } \frac{2}{3} \text{ of } x,$$

that is, $\frac{2}{3}$ of $\frac{x}{1}$ is $\frac{2x}{3}$; therefore $\frac{2x}{3} + 4$ signifies $\frac{2}{3}$ of x with 4 over; and

since this expression according to the problem, amounts to the same with

the other, to wit, $\frac{7x}{12} + 9$; hence it is that we pronounce them equal to

one another.

Now since in the foregoing equation, as well as in almost all others arising immediately from the conditions of problems themselves, the unknown quantity is embarrassed and entangled with such as are known, the way to disengage it from such known quantities, so that itself alone possessing one side of the equation, may be found equal to such as are entirely known on the other, that is, in the present case, to determine the value of the unknown quantity x , is what is commonly called the resolution of an equation: for the effecting of which, several axioms and processes are required; some whereof, namely such as most frequently occur, I shall here put down; the rest I shall take notice of occasionally, as they offer themselves.

Of the resolution of simple equations.

A X I O M I.

Whenever a fraction is to be multiplied by a whole number, it will be sufficient to multiply only the numerator by that number, retaining the denominator the same as before. Thus $\frac{4}{5}$ multiplied into 2, gives $\frac{8}{5}$, for the same reason that 4 shillings multiplied into 2 gives 8 shillings: thus in

the first example following, $\frac{7x}{12}$ multiplied into 3, gives $\frac{21x}{12}$.

A X I O M

AXIOM 2.

But if the whole number into which the fraction is to be multiplied, be equal to the denominator of the fraction, then throw away the denominator, and the numerator alone will be the product. Thus the fraction $\frac{a}{b}$ multi-

plied into b , gives $\frac{ab}{b}$ or a : thus in the first example, $\frac{2x}{3}$ multiplied into 3, gives $2x$; and $\frac{21x}{12}$ multiplied into 12 gives $21x$.

AXIOM 3.

If the two sides of an equation be multiplied or divided by the same number, the two products, or quotients, will still be equal to each other.

Thus in the first example, where $\frac{2x}{3} + 4 = \frac{7x}{12} + 9$; if both sides of

the equation be multiplied into 3, we shall have $2x + 12 = \frac{21x}{12} + 27$; and if again this last equation be multiplied into 12, we shall have $24x + 144 = 21x + 324$.

AXIOM 4.

If a quantity be taken from either side of an equation, and placed on the other with a contrary sign, which is commonly called transposition, the two sides will still be equal to each other. Thus if $7 + 3 = 10$, transpose $+3$, and you will have $7 = 10 - 3$; thus if $7 - 3 = 4$, transpose -3 , and you will have $7 = 4 + 3$; thus if (as in the first example) $24x + 144 = 21x + 324$, transpose $21x$, and you will have $24x - 21x + 144 = 324$, that is, $3x + 144 = 324$; and if again in this last equation you transpose 144, you will have $3x = 324 - 144 = 180$.

Transposition therefore, as it is here delivered, is nothing but a general name for adding or subtracting equal quantities from the two sides of an equation; in which case it is no wonder, if the sums or differences still continue equal to each other. As for instance, in this equation $a - b = c$, transposing $-b$ we have $a = c + b$: and what is this after all, but adding b to both sides of the equation? for if b be added to $a - b$, the sum will be a ; and if b be added to c , the sum will be $c + b$; therefore $a = c + b$: again, in the equation $a + b = c$, transposing $+b$ we have $a = c - b$, which is nothing else but subtracting b from both sides of the equation.

The

The 1st Process.

If, when an equation is to be resolved, fractions be found on one, or both sides, it must be freed from them by multiplying the whole equation into the denominators of those fractions successively.

The 2d Process.

After the equation is thus reduced to integral terms, if the unknown quantity be found on both sides the equation, let it be brought by transposition to one and the same side, viz. to that side, which after reduction will exhibit it affirmative.

The 3d Process.

After this, if any loose known quantities be found on the same side with the unknown, let them also be brought by transposition to the other side of the equation.

The 4th Process.

If now the unknown quantity has any coefficient before it, divide all by that coefficient, and the equation will be resolved.

The 5th Process.

If the whole equation can be divided by the unknown quantity, let such a division be made, and the equation will be reduced to a more simple one. Thus in the 16th example you have $615x - 7xxx = 48x$; divide the whole equation by x , and you will have $615 - 7xx = 48$. In the 13th example you have $\frac{42x}{x-2} = \frac{35x}{x-3}$; divide the whole by x , which is done by dividing only the numerators of the two fractions, and you will have $\frac{42}{x-2} = \frac{35}{x-3}$.

The 6th Process.

If at last the square of the unknown quantity, and not the unknown quantity itself, appears to be equal to some known quantity on the other side of the equation, then the unknown quantity must be made equal to the square root of that which is known. Thus in the 14th example we have $xx = 36$; therefore $x = 6$, and not 18: in the 15th, we have $xx = 64$; therefore $x = 8$, the square root of 64, and not 32, it's half.

N

Examples

Examples of the resolution of simple equations.

24. This preparation being made, I shall now give some examples of the resolution of simple equations; and my first example shall be the equation given in the last article, in order to trace out the number there described.

Example 1.

$$\frac{2x}{3} + 4 = \frac{7x}{12} + 9.$$

In this equation it is plain, that there are two fractions, $\frac{2x}{3}$, and $\frac{7x}{12}$, which must be taken off at two several operations, thus: as 3 is the denominator of the first fraction, multiply the whole equation by 3, and

you will have $2x + 12 = \frac{7x}{4} + 27$: again, as the denominator of the remaining fraction is 4, multiply all by 4, and you will have $8x + 48 = 7x + 108$; which is an equation free from fractions.

2dly, It must in the next place be considered, that in this last equation $8x + 48 = 7x + 108$, the unknown quantity is concerned on both sides, to wit, $8x$ on one side, and $7x$ on the other; transpose therefore $7x$, and you will have $8x - 7x + 48 = 108$, that is, $x + 48 = 108$. If it be asked why I chose to transpose $7x$ rather than $8x$; my answer is, that had $8x$ been transposed, the unknown quantity, or it's coefficient at least, after reduction, would have been negative, contrary to the rule in the second process; for resuming the equation $8x + 48 = 7x + 108$, if $8x$ be transposed, we shall have $48 = 7x - 8x + 108$, that is, $48 = -x + 108$: but even in this case, another transposition will set all right; for if $-x$ be transposed in this last equation, we shall then have $x + 48 = 108$ as before: all that can be said then against this last way is, that it creates unnecessary transpositions, which an artist would always endeavour to avoid.

3dly, Having now reduced the equation to a much greater degree of simplicity than before, to wit, $x + 48 = 108$; because the unknown quantity x has still a loose quantity, viz. 48 joined with it, transpose that quantity 48 to the other side of the equation, and you will have $x = 108 - 48$, that is, $x = 60$.

N. B. By a loose quantity I mean such a one as is joined with the unknown by the sign $+$ or $-$, and not by way of multiplication, as is the coefficient 3 in the last equation.

4thly,

Art. 24.

SIMPLE EQUATIONS.

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4^{thly}, By this time the quantity x is very near being discovered; for if $3x = 180$, it is but dividing all by 3, and we shall have $x = 60$: 60 therefore is the number described in the last article by this property, to wit, that $\frac{2}{3}$ of it with 4 over, will amount to the same as $\frac{4}{5}$ of it with 9 over: and that 60 has this property, will now be easily made to appear synthetically; for $\frac{2}{3}$ of 60 is 40, and this with 4 over is 44; moreover $\frac{4}{5}$ of 60 is 48, and this with 9 over is also 57.

N. B. A demonstration that proves the connexion between any number and the property ascribed to it, is either analytical or synthetical: if this connexion is shewn by tracing the number from the property, the demonstration of it is called an analytical demonstration; but if it is shewn by tracing the property from the number, the demonstration is then said to be synthetical.

Example 2.

$$\frac{2x}{3} + 12 = \frac{4x}{5} + 6.$$

Here multiply by 3, and you will have $2x + 36 = \frac{12x}{5} + 18$; multiply again by 5, and you will have $10x + 180 = 12x + 90$; transpose $10x$, and you will have $180 = 12x - 10x + 90$, that is, $180 = 2x + 90$, or rather $2x + 90 = 180$, for I generally choose to have the unknown quantity on the first side of the equation: transpose 90, and you will have $2x = 180 - 90$, that is, $2x = 90$; divide by 2, and you will have $x = 45$.

The Proof.

The original equation was $\frac{2x}{3} + 12 = \frac{4x}{5} + 6$: now if $x = 45$, we have $\frac{2x}{3} = 30$, and $\frac{2x}{3} + 12 = 42$: again, we have $\frac{4x}{5} = 36$, and $\frac{4x}{5} + 6 = 42$; therefore $\frac{2x}{3} + 12 = \frac{4x}{5} + 6$, because the amount of both is 42.

Example 3.

$\frac{3x}{4} + 5 = \frac{5x}{6} + 2$: therefore $3x + 20 = \frac{20x}{6} + 8$; therefore $18x + 120 = 20x + 48$; therefore $120 = 20x - 18x + 48$, that is, $120 = 2x + 48$; therefore $120 - 48 = 2x$, that is, $2x = 72$; therefore $x = 36$.

N 2

The

The Proof.

The original equation was $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$: now if $x = 36$, we shall have $\frac{3x}{4} = 27$, and $\frac{3x}{4} + 5 = 32$: we shall also have $\frac{5x}{6} = 30$, and $\frac{5x}{6} + 2 = 32$; therefore if $x = 36$, we shall have $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$.

Example 4.

$\frac{7x}{8} - 5 = \frac{9x}{10} - 8$: therefore $7x - 40 = \frac{72x}{10} - 64$; therefore $70x - 400 = 72x - 640$; therefore $-400 = 72x - 70x - 640$, that is, $-400 = 2x - 640$, or rather $2x - 640 = -400$; therefore $2x = 640 - 400$, that is, $2x = 240$; and $x = 120$.

The Proof.

The original equation, $\frac{7x}{8} - 5 = \frac{9x}{10} - 8$; but $x = 120$; therefore $\frac{7x}{8} = 105$; therefore $\frac{7x}{8} - 5 = 100$: moreover $\frac{9x}{10} = 108$; therefore $\frac{9x}{10} - 8 = 100$; therefore $\frac{7x}{8} - 5 = \frac{9x}{10} - 8$.

Example 5.

$\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$: therefore $5x - 72 = 666 - \frac{63x}{12}$; therefore $60x - 864 = 7992 - 63x$; therefore $60x + 63x - 864 = 7992$, that is, $123x - 864 = 7992$; therefore $123x = 7992 + 864$, that is, $123x = 8856$; and $x = 72$.

The Proof.

The original equation, $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$; $x = 72$; therefore $\frac{5x}{9} = 40$; therefore $\frac{5x}{9} - 8 = 32$: again, $\frac{7x}{12} = 42$; therefore $74 - \frac{7x}{12} = 74 - 42 = 32$; therefore $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$.

Example

Example 6.

$\frac{x}{6} - 4 = 24 - \frac{x}{8}$: therefore $x - 24 = 144 - \frac{6x}{8}$; therefore $8x - 192 = 1152 - 6x$; therefore $8x + 6x - 192 = 1152$, that is, $14x - 192 = 1152$; therefore $14x = 1152 + 192$, that is, $14x = 1344$; and $x = 96$.

The Proof.

The original equation, $\frac{x}{6} - 4 = 24 - \frac{x}{8}$; $x = 96$; $\frac{x}{6} = 16$; $\frac{x}{6} - 4 = 12$; again, $\frac{x}{8} = 12$; therefore $24 - \frac{x}{8} = 24 - 12 = 12$; therefore $\frac{x}{6} - 4 = 24 - \frac{x}{8}$.

Example 7.

$56 - \frac{3x}{4} = 48 - \frac{5x}{8}$: therefore $224 - 3x = 192 - \frac{20x}{8}$; therefore $1792 - 24x = 1536 - 20x$; therefore $1792 = 1536 + 24x - 20x$, that is, $1792 = 1536 + 4x$; therefore $1792 - 1536 = 4x$, that is, $4x = 256$; and $x = 64$.

The Proof.

The original equation, $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$; $x = 64$; therefore $\frac{3x}{4} = 48$; therefore $56 - \frac{3x}{4} = 56 - 48 = 8$; again, $\frac{5x}{8} = 40$; therefore $48 - \frac{5x}{8} = 48 - 40 = 8$; therefore $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$.

Example 8.

$36 - \frac{4x}{9} = 8$: therefore $324 - 4x = 72$; therefore $324 = 72 + 4x$; therefore $324 - 72 = 4x$, that is, $4x = 252$; and $x = 63$.

The Proof.

The original equation, $36 - \frac{4x}{9} = 8$; $x = 63$; therefore $\frac{4x}{9} = 28$; therefore $36 - \frac{4x}{9} = 36 - 28 = 8$.

Example

Example 9.

$\frac{2x}{3} = \frac{176-4x}{5}$: therefore $2x = \frac{528-12x}{5}$; therefore $10x = 528 - 12x$; therefore $10x + 12x = 528$, that is, $22x = 528$; and $x = 24$.

The Proof.

The original equation, $\frac{2x}{3} = \frac{176-4x}{5}$; $x = 24$; therefore $\frac{2x}{3} = 16$: again, $4x = 96$; therefore $176 - 4x = 176 - 96 = 80$; therefore $\frac{176-4x}{5} = \frac{80}{5} = 16$; therefore $\frac{2x}{3} = \frac{176-4x}{5}$.

Example 10.

$\frac{3x}{4} + \frac{180-5x}{6} = 29$: therefore $3x + \frac{720-20x}{6} = 116$; therefore $18x + 720 - 20x = 696$, that is, $720 - 2x = 696$; therefore $720 = 2x + 696$; therefore $720 - 696 = 2x$, that is, $2x = 24$; and $x = 12$.

The Proof.

The original equation, $\frac{3x}{4} + \frac{180-5x}{6} = 29$; $x = 12$; therefore $\frac{3x}{4} = 9$; $5x = 60$; therefore $180 - 5x = 180 - 60 = 120$; therefore $\frac{180-5x}{6} = \frac{120}{6} = 20$; therefore $\frac{3x}{4} + \frac{180-5x}{6} = 29$.

Example 11.

$$\frac{45}{2x+3} = \frac{57}{4x-5}$$

Multiply by $2x+3$, and you will have $45 = \frac{114x+171}{4x-5}$; multiply by $4x-5$, and you will have $180x-225 = 114x+171$; therefore $180x-114x-225 = 171$, that is, $66x-225 = 171$; therefore $66x = 171+225$, that is, $66x = 396$; and $x = 6$.

The

The Proof.

The original equation, $\frac{45}{2x+3} = \frac{57}{4x-5}$; $x=6$; therefore $2x=12$;
 therefore $2x+3=15$; therefore $\frac{45}{2x+3} = \frac{45}{15} = 3$; again, $4x=24$;
 therefore $4x-5=19$; therefore $\frac{57}{4x-5} = \frac{57}{19} = 3$; therefore $\frac{45}{2x+3}$
 $= \frac{57}{4x-5}$.

Example 12.

$\frac{128}{3x-4} = \frac{216}{5x-6}$: therefore $128 = \frac{648x-864}{5x-6}$; therefore $640x$
 $-768 = 648x-864$; therefore $-768 = 648x-640x-864$,
 that is, $-768 = 8x-864$; therefore $+864-768 = 8x$, that is,
 $8x = 96$; and $x = 12$.

The Proof.

The original equation, $\frac{128}{3x-4} = \frac{216}{5x-6}$; $x=12$; therefore $3x$
 $= 36$; therefore $3x-4=32$; therefore $\frac{128}{3x-4} = \frac{128}{32} = 4$; again,
 $5x=60$; therefore $5x-6=54$; therefore $\frac{216}{5x-6} = \frac{216}{54} = 4$;
 therefore $\frac{128}{3x-4} = \frac{216}{5x-6}$.

Example 13.

$\frac{42x}{x-2} = \frac{35x}{x-3}$: divide both numerators by x , and you will have
 $\frac{42}{x-2} = \frac{35}{x-3}$; therefore $42 = \frac{35x-70}{x-3}$; therefore $42x-126 =$
 $35x-70$; therefore $42x-35x-126 = -70$, that is, $7x-126$
 $= -70$; therefore $7x = 126-70$, that is, $7x = 56$; and $x = 8$.

The Proof.

The original equation, $\frac{42x}{x-2} = \frac{35x}{x-3}$; $x=8$; therefore $x-2=6$;

42x

$42x = 336$; therefore $\frac{42x}{x-2} = \frac{336}{6} = 56$: again, $x-3 = 5$; and
 $35x = 280$; therefore $\frac{35x}{x-3} = \frac{280}{5} = 56$; therefore $\frac{42x}{x-2} = \frac{35x}{x-3}$.

Example 14.

$\frac{xx-12}{3} = \frac{xx-4}{4}$: therefore $xx-12 = \frac{3xx-12}{4}$; therefore
 $4xx-48 = 3xx-12$; therefore $4xx-3xx-48 = -12$, that
 is, $xx-48 = -12$; therefore $xx = +48-12$, that is, $xx = 36$;
 and $x = 6$.

The Proof.

The original equation, $\frac{xx-12}{3} = \frac{xx-4}{4}$; $x = 6$; therefore $xx =$
 36 ; therefore $xx-12 = 24$; therefore $\frac{xx-12}{3} = \frac{24}{3} = 8$: again, xx
 $-4 = 32$; therefore $\frac{xx-4}{4} = \frac{32}{4} = 8$; therefore $\frac{xx-12}{3} = \frac{x^2-4}{4}$.

Example 15.

$\frac{5xx}{16} - 8 = 12$: therefore $5xx-128 = 192$; therefore $5xx = 192$
 $+ 128$, that is, $5xx = 320$; therefore $xx = 64$; and $x = 8$.

The Proof.

The original equation, $\frac{5xx}{16} - 8 = 12$; $x = 8$; therefore $xx = 64$;
 therefore $5xx = 320$; therefore $\frac{5xx}{16} = \frac{320}{16} = 20$; therefore $\frac{5xx}{16} -$
 $8 = 20 - 8 = 12$.

Example 16.

$615x - 7xxx = 48x$: divide the whole by x , and you will have
 $615 - 7xx = 48$; therefore $615 = 7xx + 48$; therefore $615 - 48$
 $= 7xx$, that is, $7xx = 567$; therefore, $xx = 81$; and $x = 9$.

The Proof.

The original equation, $615x - 7xxx = 48x$; $x = 9$; therefore
 $xx = 81$; therefore $xxx = 729$; $7xxx = 5103$; again, $615x =$
 5535 ; therefore $615x - 7xxx = 5535 - 5103 = 432$: lastly, $48x =$
 432 ; therefore $615x - 7xxx = 48x$.

T H E

T H E

ELEMENTS of ALGEBRA

B O O K H.

Preparations for the solution of Algebraic problems.

Article 25. **I**N solving the following problems, I shall make use of a sort of mixt Algebra, using letters only in representing unknown quantities, and numbers for such as are known. This method, as I take it, will be the best to begin with: but afterwards, when my young scholar has been sufficiently exercised in this way, I shall then introduce him into pure Algebra, which he will find much more extensive than the former, not only as it enables him analytically to find out general solutions, taking in all the particular cases that can be proposed in the problem to which the solution belongs, but also as it enables him afterwards to demonstrate the same solutions or theorems synthetically.

And because I am not yet to suppose him skilled in any of the mathematical sciences, I shall draw my problems, generally speaking, from numbers, either considered abstractedly, or else as they relate to common life.

If a problem be justly proposed, it ought to have as many independent conditions comprehended in it, expressly or implicitly, as there are unknown quantities to be discovered by them; and it must be the chief business of an Algebrist, to search out, sift and distinguish these conditions one from another, before ever he enters upon the solution of his problem.

I said, that so many conditions ought to be comprehended in the problem expressly or implicitly, because it may happen, that a condition may not be expressed in a problem, and yet be implied in the nature of the thing: thus in the 44th problem, where several rods are to be set upright in a straight line, at certain intervals one from another, it is implied, though not expressed, that the number of intervals must be less than the number of rods by unity.

Sometimes a condition may be introduced into a problem, that includes two or more conditions: as when we say, four numbers are in continual proportion, we mean, not only that the first number is to the second as the second is to the third, but also, that the second number is to the third as the third is to the fourth.

Whenever a problem is proposed to be solved algebraically, the Algebraist must substitute some letter of the alphabet for the unknown quantity; and if there be more unknown quantities than one, the rest must receive their names from so many conditions of the problem; and if the problem be justly stated and examined, there will still remain a condition at last, which translated into algebraic language, will afford him an equation, the resolution whereof will give the unknown quantity for which the substitution was made; and when this unknown quantity is once discovered, the rest will be easily discovered by their names. Suppose there are four unknown quantities in a problem; then there ought to be four conditions: now the first unknown quantity receives its name arbitrarily without any condition; therefore the other three must take up three of the conditions of the problem for their names; and the fourth condition will still be left to furnish out an equation.

The learner must here be very careful to make no positions but what are sufficiently justifiable, either from the express conditions of the problem, or from the nature of the thing; all the liberty he is allowed in cases of this nature is, that he is not obliged to draw out the conditions in the same order as they are given him in the problem, but may make use of them in such an order, as he thinks will be most convenient for his purpose; provided that he does not make use of the same condition twice, except in company with others that have not yet been considered.

My method in the forty four following problems will be, to put down the answer immediately after the problem, and then the solution: for in my opinion, this way of putting down the answer first, will not only serve to illustrate the following solution, but may also serve to fix the problem more firmly in the minds of young beginners, who are but too apt to neglect it, and to substitute chimerical notions of their own, that are not to be justified, either from the conditions of the problem, or common sense.

After the learner has run over some of these problems, and has got a tolerable insight into the method of their resolution, it will be very proper for him to begin again, and to attempt the solution of every problem himself, and not to have recourse to the solutions here given, but in cases of absolute necessity: but after the work is over, he may then compare his own solution with that which is here given, and may alter or reform it as he thinks fit.

The

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The solution of some problems producing simple equations.

PROBLEM 1.

26. *What two numbers are those, whose difference is 14, and whose sum when added together, is 48?*

Ans. The numbers are 31 and 17: for $31 - 17 = 14$; and $31 + 17 = 48$.

SOLUTION.

In this problem there are two unknown quantities, to wit, the two numbers sought; and there are two conditions; first, that the less number when subtracted from the greater must leave 14; and secondly, that the two numbers when added together must make 48: therefore I put x for the less number; and to find a name for the greater, I have recourse to the first condition of the problem, which informs me, that the difference betwixt the two numbers sought is 14; therefore, if I call the less number x , I ought to call the greater $x + 14$: thus then I have got names for both my unknown quantities, and have still a condition in reserve for an equation, which is the second: now according to this second condition, the two numbers sought, when added together, must make 48; therefore x and $x + 14$ when added together must make 48; but x and $x + 14$ when added together make $2x + 14$; whence I have this equation, $2x + 14 = 48$; therefore $2x = 48 - 14 = 34$; therefore x , or the less number $= 17$, and $x + 14$, or the greater number $= 31$, as above.

In our solution of this problem, the notation was drawn from the first condition, and the equation from the second; but the notation might have been drawn from the second condition, and the equation from the first, thus: put x for the less number sought; then because the sum of both the numbers is 48, if you subtract the less number x from 48, the remainder $48 - x$ will be the greater number, so that the two numbers sought will be x , and $48 - x$; subtract the former number from the latter, and the remainder or difference will be $48 - 2x$; but according to the first condition of the problem, this difference ought to be 14; therefore $48 - 2x = 14$: resolve this equation, and you will have $x = 17$, and $48 - x = 31$, as above.

PROBLEM 2.

27. *Three persons, A, B and C, make a joint contribution, which in the whole amounts to 76 pounds: of this, A contributes a certain sum unknown; B contributes as much as A, and 10 pounds more; and C, as*

much as both A and B together : I demand their several contributions.

Ans. A contributes 14 pounds, B 24, and C 38: for $14+10=24$, and $14+24=38$, and $14+24+38=76$.

SOLUTION.

In this problem there are three unknown quantities, and there are three conditions for finding them out; first, that the whole contribution amounts to 76 pounds; secondly, that B contributes as much as A, and 10 pounds more; and lastly, that C contributes as much as both A and B together.

These things being supposed, I first put x for A's contribution; then since, according to the second condition, B contributes as much as A, and 10 pounds more, I put $x+10$ for B's contribution; lastly, since C contributes as much as both A and B together, I add x and $x+10$ together, and so put down the sum $2x+10$ for C's contribution: thus have I got names for all my unknown quantities, and there remains still one condition unconsidered for my equation, which is, that all the contributions added together make 76 pounds; therefore I add x , and $x+10$, and $2x+10$ together, and suppose the sum $4x+20=76$; therefore $4x=76-20=56$; therefore x , or A's contribution equals 14; $x+10$, or B's contribution equals 24; and $2x+10$, or C's contribution equals 38, as above.

PROBLEM 3.

28. *Suppose all things as before, except that now, the whole contribution amounts to 276 pounds; that of this, A contributes a certain sum unknown; that B contributes twice as much as A, and 12 pounds more; and C three times as much as B, and 12 pounds more: I demand their several contributions.*

Ans. A contributes 24 pounds, B 60, and C 192: for $24 \times 2 + 12 = 60$; and $60 \times 3 + 12 = 192$; and $24 + 60 + 192 = 276$.

SOLUTION.

Put x for A's contribution; then because B contributes twice as much as A, and 12 pounds more, B's contribution will be $2x+12$; therefore if C had contributed just three times as much as B, his contribution would have amounted to $6x+36$; but according to the problem, C contributes this, and 12 pounds more; therefore C's contribution is $6x+48$; add these contributions together, to wit, x , $2x+12$, and $6x+48$, and you will have $9x+60=276$; therefore $9x=276-60=216$; and x , or A's contribution equals 24; whence $2x+12$, or B's

Art. 28, 29. PRODUCING SIMPLE EQUATIONS. 109
B's contribution equals 60; and $6x+48$, or *C's* contribution equals 192, as above.

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I know not whether it may not be thought impertinent here to put the learner in mind, that after x was found equal to 24, the other two unknown quantities, $2x+12$, and $6x+48$ were found, by substituting 24 instead of x .

PROBLEM 4.

29. *One begins the world with a certain sum of money, which he improved so well by way of traffick, that at the years end, he found he had doubled his first stock, except an hundred pounds laid out in common expences; and so he went on every year doubling the last years stock, except a hundred a year expended as before; and at the end of three years, found himself just three times as rich as at first: What was his first stock?*

Ans. 140 pounds: for the double of this is 280, and $280-100=180$ pounds at the end of the first year; the double of this last is 360, and $360-100=260$ pounds at the end of the second year; again, the double of this is 520, and $520-100=420$ pounds at the end of the third year; and 420 pounds is just three times as much as 140 pounds, his first stock.

SOLUTION.

Put x for his first stock, that is, let x be the number of pounds he began with; then the double of this is $2x$, and therefore he will have $2x-100$ at the end of the first year; the double of this is $4x-200$; therefore he will have $4x-200-100$, that is, $4x-300$ at the end of the second year; the double of this is $8x-600$; therefore he will have $8x-600-100$, that is, $8x-700$ at the end of the third year; but according to the problem, he ought to have three times his first stock, that is, $3x$, at the end of the third year; therefore $8x-700=3x$; therefore $8x-3x-700=0$, that is, $5x-700=0$; therefore $5x=700$; and x , or his first stock equals 140, as above.

To this problem I shall add another of a like kind, for the learner to solve himself.

One goes with a certain quantity of money about him to a tavern, where he borrows as much as he had then about him, and out of the whole, spends a shilling; with the remainder he goes to a second tavern, where he borrows as much as he had then left, and there also spends a shilling; and so he goes on to a third, and a fourth tavern, borrowing and spending

ing as before ; after which he had nothing left : I demand how much money he had at first about him.

Ans. $\frac{15}{16}$ of one shilling, that is, 11 pence farthing.

PROBLEM 5.

30. *One has six sons, each whereof is four years older than his next younger brother ; and the eldest is three times as old as the youngest : What are their several ages ?*

Ans. 10, 14, 18, 22, 26, 30: for 30, the age of the eldest, will then be just three times 10, that is, three times the age of the youngest.

SOLUTION.

For their several ages put x , $x+4$, $x+8$, $x+12$, $x+16$, $x+20$; then according to the problem $x+20$ the age of the eldest, ought to be equal to $3x$, that is, three times the age of the youngest; since then $3x = x+20$, we shall have $3x - x = 20$, that is, $2x = 20$, and $x = 10$, as above.

PROBLEM 6.

31. *There is a certain fish whose head is 9 inches ; the tail is as long as the head and half the back ; and the back is as long as both the head and tail together : I demand the length of the back, and of the tail.*

Ans. The length of the back was 36 inches, and that of the tail 27: for $27 = 9 + \frac{16}{2}$; and $36 = 9 + 27$.

SOLUTION.

For the length of the back put x ; then will x be equal to the length of both head and tail together, by the supposition; therefore if from x , the length of the head and tail together, you subtract 9, the length of the head, there will remain $x-9$ for the length of the tail; but according to the problem, the tail is as long as the head and half the back; therefore $x-9 = \frac{x}{2} + 9$; therefore $2x - 18 = x + 18$; therefore $2x - x - 18 = 18$, that is, $x - 18 = 18$; and x , the length of the back equals $18 + 18 = 36$; therefore $x - 9$, the length of the tail equals 27, as above.

PROBLEM 7.

32. *One has a lease for 99 years ; and being asked how much of it was already expired, answered, that two thirds of the time past was equal to four fifths of the time to come : I demand the times past, and to come.*

Ans.

Art. 32, 33. PRODUCING SIMPLE EQUATIONS. IIX

Ans. The time past was 54 years; and the whole term of years was 99; therefore the time to the expiration of the lease was 45 years: now $\frac{2}{3}$ of 54 is 36; and $\frac{4}{5}$ of 45 is 36.

SOLUTION.

Put x for the time past; then since the whole term of years was 99, if x the time past, be subtracted from 99 the whole time, there will remain $99 - x$ for the time to come; but $\frac{2}{3}$ of the time past is $\frac{2x}{3}$; and $\frac{4}{5}$ of the time to come is $\frac{4}{5}$ of $\frac{99 - x}{1} = \frac{396 - 4x}{5}$; therefore $\frac{2x}{3} = \frac{396 - 4x}{5}$; therefore $2x = \frac{1188 - 12x}{5}$; therefore $10x = 1188 - 12x$; therefore $10x + 12x = 1188$, that is, $22x = 1188$; and x the time past = 54 years; therefore $99 - x$ the time to come equals 45 years.

To this problem I shall add two others of the same nature, without any solution.

First, *To divide the number 84 into two such parts, that three times one part may be equal to four times the other.*

Ans. The parts are 48 and 36: for in the first place, $48 + 36 = 84$; and in the next place, three times 48 = 144 = four times 36.

Second, *To divide the number 60 into two such parts, that a seventh part of one may be equal to an eighth part of the other.*

Ans. The parts are 28 and 32: for in the first place, $28 + 32 = 60$; and in the next place, $\frac{1}{7}$ of 28 equals 4 = $\frac{1}{8}$ of 32.

PROBLEM 8.

33. *It is required to divide the number 50 into two such parts, that $\frac{3}{4}$ of one part being added to $\frac{5}{6}$ of the other, may make 40.*

Ans. The parts are 20 and 30: for in the first place, $20 + 30 = 50$; and in the next place, $\frac{3}{4}$ of 20, which is 15, added to $\frac{5}{6}$ of 30, which is 25, makes 40.

SOLUTION.

Put x for one part, and consequently $50 - x$ for the other part; then we shall have $\frac{3}{4}$ of $x = \frac{3x}{4}$, and $\frac{5}{6}$ of $50 - x = \frac{250 - 5x}{6}$; but according to the problem, these two added together ought to make 40; whence we have this equation, $\frac{3x}{4} + \frac{250 - 5x}{6} = 40$: multiply by 4, and you will

will have $3x + \frac{1000 - 20x}{6} = 160$; multiply again by 6, and you will have $18x + 1000 - 20x = 960$, that is, $1000 - 2x = 960$; therefore $1000 = 2x + 960$; and $1000 - 960 = 2x$, that is, $2x = 40$; and x , which is one of the parts sought, will be 20; whence $50 - x$ or the other part will be 30, as above.

Other two problems of the same nature.

First: *It is required to divide the number 20 into two such parts, that three times one part being added to five times the other may make 84.*

Ans. The parts are 8 and 12: for $8 + 12 = 20$; and $8 \times 3 + 12 \times 5$, that is, $24 + 60 = 84$.

Second: *It is required to divide the number 100 into two such parts, that if a third part of one be subtracted from a fourth part of the other, the remainder may be 11.*

Ans. The parts are 24 and 76: for first, 24 added to 76 makes 100; and secondly, $\frac{1}{3}$ part of 24, which is 8, subtracted from $\frac{1}{4}$ of 76, which is 19, leaves 11.

PROBLEM 9.

34. *Two persons A and B engage at play; A has 72 guineas and B 52 before they begin; and after a certain number of games won and lost between them, A rises with three times as many guineas as B: I demand how many guineas A won of B.*

Ans. 21: for $72 + 21 = 93$; and $52 - 21 = 31$; and $93 = 3 \times 31$.

SOLUTION.

Put x for the number of guineas A won of B , and consequently that B lost; then will A 's last sum be $72 + x$, and B 's last sum $52 - x$: now according to the problem, A 's last sum is three times as much as B 's last sum; that is, three times $52 - x$, or $156 - 3x$; therefore $72 + x = 156 - 3x$; therefore $72 + x + 3x = 156$, that is, $72 + 4x = 156$; therefore $4x = 156 - 72 = 84$; therefore x , the money A won of B , equals 21 guineas, as above.

PROBLEM 10.

35. *One meeting a company of beggars, gives to each four pence, and has sixteen pence over; but if he would have given them six pence apiece, he would have wanted twelve pence for that purpose: I demand the number of persons.*

Ans. 14: for $14 \times 4 + 16 = 72 = 14 \times 6 - 12$.

SOLU-

Art. 35, 36, 37. PRODUCING SIMPLE EQUATIONS. 113

SOLUTION.

Put x for the number of persons; then if he gives them four pence apiece, the number of pence given will be four times as many as the number of persons, that is, $4x$; therefore $4x+16$ will express all the money he had about him; and so also will $6x-12$ by a like way of reasoning; therefore $4x+16=6x-12$; therefore $16=6x-4x-12=2x-12$; therefore $2x=16+12=28$; and x , the number of persons equals 14, as above.

PROBLEM II.

36. *What two numbers are those, whose difference is 4, and the difference of whose squares is 112?*

Ans. 12 and 16: for $16-12=4$, and $16 \times 16-12 \times 12$, that is, $256-144=112$.

SOLUTION.

The less number, x .	$x+4$
The greater, $x+4$.	$x+4$
	<hr/>
	$xx+4x+16$
	$+4x$
	<hr/>
The square of the greater,	$xx+8x+16$
The square of the less,	xx
	<hr/>

The difference of their squares, $* 8x+16$; whence $8x+16=112$; therefore $8x=112-16=96$; therefore x the less number equals 12, and $x+4$ the greater equals 16, as above.

PROBLEM 12.

37. *What two numbers are those, whereof the greater is three times the less, and the sum of whose squares is five times the sum of the numbers?*

Ans. The numbers are 6 and 2, whose sum is 8: now $6=3$ times 2; and $6 \times 6+2 \times 2=40=5$ times 8.

SOLUTION.

The less number,	x .
The greater,	$3x$.
Their sum,	$4x$.
The square of the less,	xx .
The square of the greater,	$9xx$.
The sum of their squares,	$10xx$.

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But according to the problem, the sum of their squares is 5 times the sum of the numbers, that is, 5 times $4x$ or $20x$; therefore $10xx = 20x$; and $10x = 20$; and x the less number $= 2$; whence $3x$ the greater equals 6, as above.

PROBLEM 13.

38. *What two numbers are those, whereof the less is to the greater as 2 to 3, and the product of whose multiplication is 6 times the sum of the numbers?*

Ans. The numbers are 10 and 15, whose sum is 25: for 10 is to 15 as 2 to 3; this will be plain by putting the question thus; if 2 gives 3, what will 10 give? for the answer will be 15: these numbers will also answer the second condition of the problem; for $10 \times 15 = 150 = 25 \times 6$.

SOLUTION.

Put x for the less number; then to find the greater number say, if 2 gives 3, what will x give? and the answer is $\frac{3x}{2}$; therefore if x stands for the less number, the greater number will be $\frac{3x}{2}$; their sum will be $\frac{x}{1} + \frac{3x}{2}$, or $\frac{2x + 3x}{2}$, or $\frac{5x}{2}$; and the product of their multiplication $x \times \frac{3x}{2}$, or $\frac{3xx}{2}$; but according to the problem, the product of their multiplication ought to be six times the sum of the numbers, that is, six times $\frac{5x}{2}$, or $\frac{30x}{2}$; therefore $\frac{3xx}{2} = \frac{30x}{2}$; and $3x^2 = 30x$; and $3x = 30$; and x the less number equals 10; therefore $\frac{3x}{2}$ the greater number equals 15, as above.

PROBLEM 14.

39. *Two persons A and B were talking of their money; says A to B, give me five shillings of your money, and I shall have just as much as you will have left: says B to A, rather give me five shillings of your money, and I shall then have just three times as much as you will have left: How much money had each?*

Ans. A had 15 shillings, and B 25: for then, if A borrows 5 shillings of B, they will have 20 shillings each; on the other hand, if

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Art. 39, 40. PRODUCING SIMPLE EQUATIONS. 115
A lends *B* 5 shillings, then will *A* have 10 shillings left, and *B* will have 30, which is three times as much.

SOLUTION.

Put x for *A*'s money; then if *A* borrows five shillings of *B*, *A* will have $x + 5$, and *B* by the supposition, will have the same left, to wit, $x + 5$; but if *B* after having lent *A* 5 shillings, has $x + 5$ left, he must have had $x + 10$ before; therefore if x represents *A*'s money, $x + 10$ will represent *B*'s: let us now suppose *B* to borrow 5 shillings of *A*; then will *B* have $x + 15$, and *A* will have $x - 5$; but according to the problem, *B* in this case ought to have three times as much as *A* has left, that is, three times $x - 5$, or $3x - 15$; therefore $3x - 15 = x + 15$; therefore $3x - x - 15 = 15$, that is, $2x - 15 = 15$; therefore $2x = 15 + 15 = 30$; therefore x , or *A*'s money equals 15 shillings, and $x + 10$, or *B*'s = 25, as above.

PROBLEM 15.

40. What two numbers are those, the product of whose multiplication is 108, and whose sum is equal to twice their difference?

Ans. 18 and 6: for the product of their multiplication is 108; and their sum 24, is equal to twice their difference 12.

SOLUTION.

For the greater number I put x ; then, had their sum been 108, I should for the other number have put $108 - x$; but it is not the sum of their addition, but the product of their multiplication that is equal to 108; therefore if one number be called x , the other will be $\frac{108}{x}$, which I thus demonstrate: let y be the other number; then will $x \times y$ or $xy = 108$ by the supposition; divide both sides of the equation by x , and you will have $y = \frac{108}{x}$; as was to be demonstrated. This being admitted,

the difference between the greater number x , and the less $\frac{108}{x}$, is $x - \frac{108}{x}$; and their sum is $x + \frac{108}{x}$: but by the condition of the problem, this sum ought to be equal to twice the difference, that is, to twice $x - \frac{108}{x}$ or $2x - \frac{216}{x}$; therefore $2x - \frac{216}{x} = x + \frac{108}{x}$; therefore $2xx$

$-216 = xx + 108$; therefore $2xx - xx - 216 = 108$, that is, $xx - 216 = 108$; therefore $xx = 108 + 216 = 324$; therefore x the greater number equals 18, and $\frac{108}{x}$ the less equals 6, as above.

PROBLEM 16.

41. *It is required to divide the number 48 into two such parts, that one part may be three times as much above 20, as the other wants of 20.*

Ans. The two parts are 32 and 16: for $32 + 16 = 48$; moreover 32 is 12 above 20, and 16 wants 4 of 20, and 12 is three times 4.

SOLUTION.

Put x for the less number sought; then will $48 - x$ be the greater, and the excess of this greater above 20 will be $28 - x$, as is evident by subtracting 20 from $48 - x$: again, the excess of 20 above the less number (which is, what the less number wants of 20) is $20 - x$; and according to the problem, the former excess is three times the latter, that is, three times $20 - x$, or $60 - 3x$; whence we have this equation, $28 - x = 60 - 3x$; therefore $28 - x + 3x = 60$, that is, $28 + 2x = 60$; therefore $2x = 60 - 28 = 32$; therefore x the less part $= 16$, and $48 - x$ the greater $= 32$, as above.

Another solution of the foregoing problem.

Put x for what the less number wants of 20; then will the less number be $20 - x$, the greater $20 + 3x$, and their sum $40 + 2x$; but by the problem, their sum is 48; therefore $40 + 2x = 48$; therefore $2x = 48 - 40 = 8$; therefore $x = 4$; whence $20 - x$ the less number $= 16$, and $20 + 3x$ the greater $= 32$.

PROBLEM 17.

42. *One has three debtors, A, B and C, whose particular debts he has forgot; but thus much he could remember from his accounts, that A's and B's debts together amounted to 60 pounds; A's and C's to 80 pounds; and B's and C's to 92 pounds: I demand the particulars.*

Ans. A's debt was 24 pounds, B's 36, and C's 56: for $24 + 36 = 60$, $24 + 56 = 80$, and $36 + 56 = 92$.

SOLUTION.

Put x for A's debt; then because A's and B's together made 60 pounds, B's debt will be $60 - x$: again, because A's and C's together made 80 pounds,

Art. 42, 43, 44. PRODUCING SIMPLE EQUATIONS. 117
 pounds, C's debt must be $80 - x$: now since according to the problem, B's and C's debts when added together make 92 pounds, I add $60 - x$, and $80 - x$ together, and suppose the sum $140 - 2x = 92$; whence $2x + 92 = 140$; and $2x = 140 - 92 = 48$; and x , that is, A's debt $= 24$ pounds; whence $60 - x$, or B's debt $= 36$ pounds; and $80 - x$, or C's, is 56 pounds, as above.

PROBLEM 18.

43. One being asked how many teeth he had remaining in his head, answered, three times as many as he had lost; and being asked how many he had lost, answered, as many as being multiplied into $\frac{1}{6}$ part of the number left, would give all he ever had at first: I demand how many he had lost, and how many he had left.

Ans. He had lost 8, and had 24 left: for then 24 the number left, will be equal to 3 times 8, the number lost; and moreover 8 the number lost, multiplied into 4, that is, into $\frac{1}{6}$ part of 24 the number left, will give $32 = 24 + 8$, all he ever had at first.

SOLUTION.

Teeth lost, x .
 left, $3x$.
 In all, $4x$.

$\frac{1}{6}$ part of the number left $\frac{3x}{6}$, or $\frac{x}{2}$; this multiplied into the number lost, makes $\frac{x}{2} \times x$ or $\frac{xx}{2}$; but according to the problem, this product is equal to all he ever had at first; whence $\frac{xx}{2} = 4x$; and $xx = 8x$; and x , the number lost $= 8$; whence $3x$, the number left $= 24$, as above.

PROBLEM 19.

44. One rents 25 acres of land at 7 pounds 12 shillings per annum; which land consists of two sorts, the better sort he rents at 8 shillings per acre, and the worse at 5: I demand the number of acres of each sort.

Ans. He had 9 acres of the better sort, and 16 of the worse: for $9 + 16$ acres $= 25$ acres; and 9 times 8 shillings $= 72$ shillings; and 16 times 5 shillings $= 80$ shillings; and $72 + 80 = 152$ shillings $= 7$ pounds 12 shillings.

SOLU-

SOLUTION.

Put x for the number of acres of the better fort; then will $25 - x$ be the number of acres of the worse fort, because both together made 25 acres: moreover, since he paid 8 shillings an acre for the better fort, he must pay 8 times as many shillings as he had acres, that is, $8x$; and since he paid 5 shillings an acre for the worse fort, he must pay 5 times as many shillings as he had acres of this fort, that is, $25 - x \times 5$, or $125 - 5x$: put both these rents together, and they will amount to $8x + 125 - 5x$, or $3x + 125$ in shillings; but they amount to 152 shillings, by the supposition; therefore $3x + 125 = 152$; therefore $3x = 152 - 125 = 27$; therefore x , the number of acres of the better fort $= 9$, and $25 - x$, the number of the worse fort $= 16$, as above.

PROBLEM 20.

45. One hires a labourer into his garden for 36 days upon the following conditions, to wit, that for every day he laboured, he was to receive two shillings and sixpence; and for every day he was absent, he was to forfeit one shilling and sixpence: now at the end of the 36 days, after due deductions made for his forfeitures, he received clear 2 pounds 18 shillings: I demand how many days he laboured, and how many he was absent.

Ans. He laboured 28 days, and loitered 8: for 28 half-crowns amount to 3 pounds 10 shillings due to him for wages; and 8 eighteenpences amount to 12 shillings due from him in forfeitures; and this latter sum subtracted from the former, leaves 2 pounds 18 shillings to be received clear.

SOLUTION.

Put x for the number of days he laboured; then will $36 - x$ represent the number of days he was absent: again, since he was to receive 30 pence for every day he laboured, the pence due to him in wages will be $30 \times x$, or $30x$; and since he was to forfeit 18 pence for every day he was absent; the pence due from him in forfeitures will be $18 \times 36 - x$, or $648 - 18x$: subtract now $648 - 18x$, the pence due from him in forfeitures, from $30x$, the pence due to him for wages; or, which is all one, add $18x - 648$ to $30x$, and there arises $48x - 648$, the pence to be received clear: but he received clear 2 pounds 18 shillings, or 696 pence, by the supposition; therefore $48x - 648 = 696$; therefore $48x = 648 + 696 = 1344$; therefore x , the number of days he laboured $= 28$; and $36 - x$, the number of days he loitered $= 8$, as above.

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Art. 46. PRODUCING SIMPLE EQUATIONS. 119

PROBLEM 21.

46. Suppose that 19 pounds of gold weigh 18 pounds in water; and moreover, that 10 pounds of silver weigh 9 pounds in water; lastly, suppose a mass weighing 106 pounds, and consisting of both gold and silver, when weighed in water, weighs only 99 pounds: I demand the distinct quantities of gold and silver in the mass.

Ans. There were 76 pounds of gold, and 30 of silver: for $76 + 30 = 106$; moreover, if 19 pounds of gold weigh 18 pounds in water, 76 pounds of gold will weigh 72 pounds in water, by the golden rule; and if 10 pounds of silver weigh 9 pounds in water, 30 pounds of silver will weigh 27 pounds in water, by the same rule; and lastly $72 + 27$, the weight of the whole mass in water $= 99$, as the problem requires.

SOLUTION.

Put x for the number of pounds of gold in the mass; then will $106 - x$ represent the number of pounds of silver; and to find the weight of the former in water, I say, if 19 pounds of gold weigh 18 pounds in water, what will x weigh? and the answer is $\frac{18x}{19}$: again, to find the weight of the silver in water, I say, if 10 pounds of silver weigh 9 pounds in water, what will $106 - x$ weigh? and the answer is $\frac{9(106 - x)}{10}$; then I add both these weights together, and their sum is $\frac{18x}{19} + \frac{9(106 - x)}{10}$; but according to the problem, the weight of the whole mass in water is 99 pounds; therefore the equation is $\frac{18x}{19} + \frac{9(106 - x)}{10} = 99$; therefore $18x + \frac{18126 - 171x}{10} = 1881$; therefore $180x + 18126 - 171x = 18810$, that is, $9x + 18126 = 18810$; therefore $9x = 18810 - 18126 = 684$; therefore x , the pounds of gold in the mass $= 76$; and $106 - x$, the pounds of silver $= 30$, as above.

Note. In the solution of this problem it is taken for granted, that all bodies of the same specific gravity, have their weights in water proportionable to their weights in air, which is easily demonstrated from hydrostatics, and is further confirmed by experiments: thus if the weight of any one quantity of gold in air be to it's weight in water as 19 to 18,

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the weight of any other quantity of gold in air will be to it's weight in water in the same proportion; and the same may be observed of silver; only as silver is a body of a different specific gravity from gold, the proportion will be different, being as 10 to 9, and sometimes as 11 to 10.

PROBLEM 22.

47. One lets out a certain sum of money at 6 per cent, simple interest; which interest in 10 years time wanted but 12 pounds of the principal: What was the principal?

Ans. The principal was 30 pounds, and the interest 18 pounds = 30—12: for as 100 pounds principal is to it's annual interest 6 pounds, so is 30 pounds principal to its annual interest 1.8 pounds; and therefore it's 10 years interest will be 18 pounds.

SOLUTION.

Put x for the number of pounds in the principal; then to find it's interest for one year, say, if 100 pounds principal give 6 pounds interest, what will x principal give? and the answer will be $\frac{6x}{100}$; this will be the interest of x for one year, and therefore it's interest for 10 years will be $\frac{60x}{100}$, or $\frac{6x}{10}$, or $\frac{3x}{5}$: but according to the problem, this interest is to be $x - 12$; for it is to want just 12 pounds of the principal, by the supposition; therefore $x - 12 = \frac{3x}{5}$; therefore $5x - 60 = 3x$; therefore $5x - 3x - 60 = 0$, that is, $2x - 60 = 0$; therefore $2x = 60$, and x the principal = 30, and $\frac{3x}{5}$ the 10 years interest = 18 pounds, as above.

PROBLEM 23.

48. One lets out 98 pounds in two different parcels; one at 5, the other at 6 per cent, simple interest; and the interest of the whole in 16 years amounted to 81 pounds: What were the two parcels?

Ans. The parcel at 5 per cent was 48 pounds, and the other at 6 per cent was 50 pounds: for in the first place, $48 + 50 = 98$; and moreover, the annual interest of 48 pounds at 5 per cent amounts to 2 pounds 8 shillings; and the annual interest of 50 pounds at 6 per cent is 3 pounds; therefore the whole interest amounts to 5 pounds 8 shillings in one year; and consequently to 81 pounds in 16 years.

SOLU-

SOLUTION.

Put x for the number of pounds in the parcel at 5 per cent, and consequently $98 - x$ for the number of pounds in the other parcel at 6 per cent; then to find the annual interest of x , say, if 100 pounds principal give 5 pounds interest, what will x give? and the answer will be $\frac{5x}{100}$; again, for the other parcel, say, if 100 pounds principal give 6 pounds interest, what will $98 - x$ give? and the answer will be $\frac{588 - 6x}{100}$; add these two interests together, to wit, $\frac{5x}{100}$ and $\frac{588 - 6x}{100}$, and the sum will be $\frac{5x + 588 - 6x}{100}$, that is, $\frac{588 - x}{100}$; this is the interest of the two parcels for one year; and therefore in 15 years time, this interest must amount to $\frac{8820 - 15x}{100}$; but it amounts to 81 pounds, by the supposition; therefore $\frac{8820 - 15x}{100} = 81$; therefore $8820 - 15x = 8100$; therefore $8820 = 15x + 8100$; therefore $15x = 8820 - 8100 = 720$; therefore x , the parcel at 5 per cent = 48 pounds; and $98 - x$, the parcel at 6 per cent = 50 pounds, as above.

PROBLEM 24.

49. *A gentleman hires a servant for a year, or 12 months, and was to allow him for his wages six pounds in money, together with a livery cloak of a certain value agreed upon: but after seven months, upon some misdemeanour of the servant, he turns him off, with the afore-said cloak and 50 shillings in money; which was all that was due to him for that time: I demand the value of the cloak.*

Ans. The value of the cloak was 48 shillings: for then his whole wages for 12 months would be 168 shillings; and by the rule of proportion, his wages for 7 months would be 98 shillings; whence subtracting 48 shillings, the value of the cloak, there would remain 50 shillings due to him in money.

SOLUTION.

Put x for the value of the cloak in shillings; then will his whole wages for 12 months be $x + 120$; and his wages for 7 months, may be found by the golden rule, saying, as 12 is to 7, so is $x + 120$ to

Q

7x

$\frac{7x+840}{12}$; but according to the problem, his wages for 7 months was the cloak and 50 shillings in money, that is, $x+50$; therefore $x+50 = \frac{7x+840}{12}$; therefore $12x+600=7x+840$; therefore $12x-7x+600=840$, that is, $5x+600=840$; therefore $5x=840-600=240$; therefore x , the value of the cloak in shillings, is 48, as above.

PROBLEM 25.

50. One distributes 20 shillings among 20 people, giving 6 pence apiece to some, and 16 pence apiece to the rest: I demand the number of persons of each denomination.

Ans. There were 8 persons who received 6 pence apiece; and 12 who received 16 pence apiece: for in the first place, $8+12=20$ persons; and since 8 sixpences are equivalent to 4 shillings, and 12 sixteen-pences to 16 shillings, we shall have in the next place, $4+16=20$ shillings.

SOLUTION.

Put x for the number persons who received 6 pence apiece; then since there were 20 persons in all, $20-x$ will be the number of those who received sixteenpence apiece: the number of pence received by the former company will be $6x$; and the number of pence received by the latter will be $20-x \times 16$, that is, $320-16x$; and therefore the whole number of pence received will be $6x+320-16x$, or $320-10x$; but according to the problem, there was received in the whole, 20 shillings, or 240 pence; therefore, $320-10x=240$; therefore $10x+240=320$; therefore $10x=320-240=80$; therefore x , the number of persons who received sixpence a piece, is 8, and consequently $20-x$, the number of the rest is 12, as above.

PROBLEM 26.

51. It is required to divide 24 shillings into 24 pieces, consisting only of ninepences and thirteenpencehalfpennies.

Ans. There must be 8 ninepences and 16 thirteenpencehalfpennies: for in the first place, $8+16=24$ pieces; and since 8 ninepences are equivalent to 6 shillings, and 16 thirteenpencehalfpennies to 18 shillings, we have in the next place $6+18=24$ shillings.

SOLU-

Art. 51, 52, 53. PRODUCING SIMPLE EQUATIONS. 123
SOLUTION.

Put x for the number of ninepences, and consequently $24 - x$ for the number of thirteenpencehalfpennies: now the number of halfpence equivalent to the former is $18x$, because there are 18 halfpence in every 9 pence; and the number of halfpence equivalent to the latter is $24 - x \times 27$, or $648 - 27x$, because there are 27 halfpence in every thirteenpencehalfpenny piece; therefore the number of halfpence equivalent to the whole will be $18x + 648 - 27x$, that is, $648 - 9x$; but according to the problem, the whole amounts to 24 shillings, or 576 halfpence; therefore $648 - 9x = 576$; therefore $9x + 576 = 648$; therefore $9x = 648 - 576 = 72$; therefore x , the number of ninepences is 8; and $24 - x$, the number of thirteenpencehalfpennies is 16, as above.

PROBLEM 27.

52. *Two persons, A and B, travelling together, A with 100, and B with 48 pounds about him, met a company of robbers, who took twice as much from A as from B, and left A thrice as much as they left B: I demand how much they took from each.*

Ans. They took 44 pounds from B, and twice as much, that is, 88 pounds from A, so they left B 4 pounds, and A 12 pounds, which is 3 times 4.

SOLUTION.

Taken from B, x .
from A, $2x$.
Left B, $48 - x$.
Left A, $100 - 2x$.

But according to the problem, they left A three times as much as they left B, that is, three times $48 - x$, or $144 - 3x$; therefore $100 - 2x = 144 - 3x$; therefore $100 - 2x + 3x = 144$, that is, $100 + x = 144$; therefore x , the sum taken from B $= 144 - 100 = 44$; and $2x$, or 88 is the sum taken from A, as above.

PROBLEM 28.

53. *A certain cistern which would be filled in 12 minutes of time by two pipes running into it, would be filled in 20 minutes by one alone; I demand in what time it would be filled by the other alone.*

Ans. In 30 minutes: for at this rate, this pipe would discharge $\frac{1}{30}$ part of a cisternful in one minute's time, and $\frac{12}{30}$ or $\frac{2}{5}$ in 12 minutes: but the other pipe in 12 minutes time discharges $\frac{12}{20}$ or $\frac{3}{5}$ of a cisternful, by the

the supposition; therefore both pipes in 12 minutes will fill $\frac{2}{3}$, that is, the whole cistern.

SOLUTION.

Put x for the time fought wherein the second pipe will fill the cistern; then to find how much of this cistern will be filled by that pipe in 12 minutes time, say by the golden rule, if in the time x be discharged one whole cisternful, how much of it will be discharged in 12 minutes? and the answer will be $\frac{12}{x}$; and for the same reason $\frac{12}{20}$ will be the part filled by the first pipe in the same time of 12 minutes; and $\frac{12}{x} + \frac{12}{20}$ will be the quantity they will both discharge in this time; but according to the problem, they ought both together to discharge one whole cisternful in this time; therefore $\frac{12}{x} + \frac{12}{20} = 1$; therefore $12 + \frac{12x}{20} = x$; therefore $240 + 12x = 20x$; therefore $20x - 12x$, or $8x = 240$; and x , the time wherein the second pipe alone would fill the cistern, is 30 minutes, as above.

PROBLEM 29.

54. *A general of an army disposing his men in rank and file in form of a square, found he had 60 men more than would stand in the figure; but when he thought of enlarging the side of his square, though but by one man more, he found he should want 41 men for that purpose: I demand his number of men, and how many they stood of a side.*

Ans. He had 2560 men, and they stood 50 of a side: for the better understanding whereof it must be observed, that soldiers are said to stand in rank and file, when a rank is placed behind the foremost, and then another behind that, and so on: thus if there be 6 ranks, and 7 men in every rank, they are said to be 7 in rank and 6 in file; and the number of men so disposed are 6 times 7, or 42. This being understood, let us now suppose the general to have 50 ranks with 50 men in every rank; then they would constitute a square whose side is 50, and the number of men so disposed would be 50 times 50, or 2500; therefore if we suppose him to have 2560 men in all, he would have 60 men more than could stand in this figure: suppose then that the side of his square was enlarged from 50 to 51 men, that is, let us suppose 51 ranks with 51 men in every rank; then it is plain, that this figure would take up 51 times 51 or 2601; but he had only 2560 men; therefore this figure would

Art. 54, 55. PRODUCING SIMPLE EQUATIONS. 125
would take up more than he had by 41; for 2560 subtracted from 2601 leaves 41.

SOLUTION.

Put x for the number of men that constituted the side of the square; then will xx or xx be the number of men the figure took up; but according to the problem, he had 60 men more; therefore $xx + 60$ will represent his whole number of men: suppose now he had $x + 1$ men of a side, then $x + 1 \times x + 1$, or $xx + 2x + 1$ will be the number of men this supposed figure would have taken up; but according to the problem, this last number $xx + 2x + 1$ must be greater than the true number of men $xx + 60$ by 41; therefore $xx + 60$ subtracted from $xx + 2x + 1$, ought to leave 41; but $xx + 60$ subtracted from $xx + 2x + 1$ leaves $+ 2x - 59$; therefore $2x - 59 = 41$; therefore $2x = 41 + 59 = 100$; therefore x , the number of men of a side $= 50$; and $xx + 60$, the whole number of men $= 2560$, as above.

$$\begin{array}{r} x + 1 \\ x + 1 \\ \hline xx + x + 1 \\ + x \\ \hline xx + 2x + 1. \\ \\ xx + 2x + 1 \\ xx \quad * + 60 \\ \hline * + 2x - 59. \end{array}$$

PROBLEM 30.

55. *There are two places 154 miles distant from each other; from whence two persons set out at the same time with a design to meet, one travelling at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours: I demand how long and how far each travelled before they met.*

Ans. As our travellers were supposed both to set out at the same time, and they must both meet at the same time, it follows, that each must perform his journey in the same time; I say then, that each performed his journey in 56 hours: for if in 2 hours the first travelled 3 miles, in 56 hours he must travel 84 miles, by the rule of proportion; in like manner, if in 4 hours the second travels 5 miles, in 56 hours he must travel 70 miles; and $84 + 70 = 154$ miles, the whole distance.

SOLUTION.

Put x for the number of hours each travelled; then to find how many miles the first travelled, say, if in 2 hours he travelled 3 miles, how far did he travel in x hours? and the answer is $\frac{3x}{2}$; then for the other say, if

if in 4 hours he travelled 5 miles, how far did he travel in x hours? and the answer is $\frac{5x}{4}$; therefore both their journeys put together make $\frac{3x}{2} + \frac{5x}{4}$; but they both travelled the whole distance, 154 miles; therefore $\frac{3x}{2} + \frac{5x}{4} = 154$; therefore $3x + \frac{10x}{4} = 308$; therefore $12x + 10x$, that is, $22x = 1232$; therefore x , the number of hours each travelled $= 56$; therefore $\frac{3x}{2}$, the number of miles the first travelled $= 84$; and $\frac{5x}{4}$, the number of miles the second travelled $= 70$, as above.

PROBLEM 31.

56. One sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours after, another sets out from the same place, and travels the same road at the rate of 5 miles in 3 hours: I demand how long and how far the first must travel before he is overtaken by the second.

Ans. The first must travel 50 hours and consequently 70 miles; the second must travel 50—8, or 42 hours, and consequently also 70 miles: since then they both set out from the same place, and the second traveller has now travelled as far as the first, he must have overtaken the first.

SOLUTION.

Put x for the number of hours the first travelled, and consequently $x - 8$ for the number of hours wherein the second travelled: then to find the miles travelled by the first, say, if in 5 hours he travels 7 miles, how far will he travel in x hours? and the answer is $\frac{7x}{5}$; then for the other say, if in 3 hours he travelled 5 miles, how far will he travel in $x - 8$ hours, and the answer is $\frac{5x - 40}{3}$; but as these two travellers both set out from the same place, and must come together at the same place, it follows, that they must both travel the same length of space; therefore $\frac{5x - 40}{3} = \frac{7x}{5}$; therefore $5x - 40 = \frac{21x}{5}$; therefore $25x - 200 = 21x$; therefore $25x - 21x - 200 = 0$, that is, $4x - 200 = 0$; therefore $4x = 200$; and x , the hours travelled by the first $= 50$; whence

Art. 56, 57. PRODUCING SIMPLE EQUATIONS. 127

whence $x - 8$, the hours travelled by the second $= 42$; $\frac{7x}{5}$, the miles travelled by the first $= 70$; and $\frac{5x - 40}{3}$, the miles travelled by the second $= 70$, as above.

PROBLEM 32.

57. One looking upon a clock, and being asked what a clock it was, answered, between 5 and 6; but a more particular answer being desired, he said, that at that very moment, the minute hand and the hour hand were together: what was the time of the day?

Ans. Just $\frac{5}{11}$ of an hour, or 27 minutes and about 16 seconds past 5.

For a clearer and more distinct explication of this answer, and of the following solution, let us call that point upon the dial-plate wherein the two hands are supposed to exist together at the instant sought, between the hours of 5 and 6, A ; then, since the hour hand moves from 12 to 5 in 5 hours, if the time of $\frac{5}{11}$ of an hour be rightly assigned, it ought to move from 12 to A in 5 hours and $\frac{5}{11}$ of an hour: again, at 5 a clock the minute hand pointed at 12, and according to this account, $\frac{5}{11}$ of an hour after, it pointed at A ; therefore the minute hand moves from 12 to A in $\frac{5}{11}$ of an hour; this then being the case, to wit, that the hour hand moves from 12 to A , in 5 hours and $\frac{5}{11}$ of an hour, and that the minute hand passes over the same space in $\frac{5}{11}$ of an hour, let us in the next place enquire how these times agree with the known velocities of these two hands; for the minute hand makes a revolution in an hour's time, and the hour hand in 12 hours; therefore the hour hand moves 12 times slower than the minute hand; therefore the hour hand ought to be 12 times as long in passing over any given space, as the minute hand is in passing through the same space; but according to our account, the hour hand passed from 12 to A in 5 hours and $\frac{5}{11}$ of an hour, and the minute hand passed over the same space in $\frac{5}{11}$ of an hour; therefore if the time enquired after be truly assigned, 5 hours and $\frac{5}{11}$ of an hour ought to be 12 times as much as $\frac{5}{11}$ of an hour; and so we find it; for $12 \text{ times } \frac{5}{11} = \frac{60}{11} = 5 \frac{5}{11}$.

SOLUTION.

Put x for the part or parts of an hour from 5 a clock to the instant sought; then will the hour hand move from 12 to A , in $x + 5$ hours, and the minute hand will pass over the same space in the time x ; and $x + 5$ will be to x as 12 to 1; all which are manifest from what has been said above; multiply the extremes and middle terms of this last proportion

proportion together, according to the 15th article, and you will have $12x = x + 5$ hours; and $12x - x$, that is, $11x = 5$ hours; and $x = \frac{5}{11}$ of an hour, as above.

PROBLEM 33.

58. *A clock has two hands turning upon the same center, whereof the swifter makes a revolution every 12 hours, and the slower every 16 hours: I demand the synodical period of these two hands.*

Ans. 48 hours.

By a synodical period is here meant, the time that passes from the instant the two hands are together, to the instant they next come together again: but to acquire a more distinct idea of this time, that is, such a one as we may form a calculation upon, let us suppose the two hands together; then it is plain, that the swifter hand will immediately get before the slower; and in time, will have got a quarter of a circle, and half a circle, and three quarters of a circle, and at last an entire circle before the slower, in which case the two hands will now be together again; and this will be the first time of their coming together again since the supposed time of their setting out; therefore a synodical period may now be defined to be the time wherein the swifter hand gets an entire circle of the slower, or makes one revolution more than the slower: since then in 48 hours the slower hand makes 3 revolutions, and the swifter 4, it follows, that 48 hours must be their true synodical period.

SOLUTION.

Put x for the number of hours in a synodical period; then to find the number of revolutions made by the slower hand in the time x , say, if in 16 hours the slower hand makes one revolution, how many will it make in the time x ? and the answer will be $\frac{x}{16}$; then for the swifter hand say, if in 12 hours it makes one revolution, how many will it make in the time x ? and the answer will be $\frac{x}{12}$; therefore $\frac{x}{16}$ is the number of revolutions made by the slower hand, and $\frac{x}{12}$ the number of revolutions made by the swifter hand in the same time x : now to know how much the swifter hand has got of the slower, I subtract the former space $\frac{x}{16}$, from the latter $\frac{x}{12}$, and the remainder is $\frac{x}{12} - \frac{x}{16}$, or $\frac{x}{48}$; but as x is the time of a synodical period, it is plain from what has been said, that the swifter hand ought to have got one entire circle or revolution

Art. 58, 59, 60. PRODUCING SIMPLE EQUATIONS. 129

lution before the flower; therefore $\frac{x}{48} = 1$; and x , the number of hours in a synodical period $= 48$, as above.

PROBLEM 34.

59. A vintner has two sorts of wines, one sort worth 20 pence a quart, and the other worth 12 pence a quart; out of which he wants a mixture of a 100 quarts worth 14 pence the quart: the question is, how many quarts he must mix of each sort.

Ans. There must be 25 quarts of the better sort, and 75, that is, $100 - 25$ of the worse: for 25 quarts at 20 pence a quart amount to 500 pence; and 75 quarts at 12 pence a quart amount to 900 pence; therefore the whole amounts to 1400 pence; which divided by 100, the number of quarts, gives 14 pence a quart.

SOLUTION.

Put x for the number of quarts of the better sort, and consequently $100 - x$ for those of the worse; then will $20x$ be the price in pence of the better sort, $1200 - 12x$ that of the worse, and $8x + 1200$ that of the whole: but 100 quarts at 14 pence a quart amount to 1400 pence; therefore $8x + 1200 = 1400$; $8x = 200$; and x , the number of quarts of the better sort $= 25$; and $100 - x$, the number of quarts of the worse sort $= 75$, as above.

PROBLEM 35.

60. It is required to divide the number 90 into 4 such parts, that the first part increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, may be all equal.

Ans. The parts are 18, 22, 10, and 40: for $18 + 22 + 10 + 40 = 90$; and $18 + 2 = 22 - 2 = 10 \times 2 = \frac{40}{2} = 20$.

SOLUTION.

1st, Put x for the first part; then if this first part be increased by 2, the sum will be $x + 2$; therefore the second part when diminished by 2, the third when multiplied by 2, and the fourth when divided by 2, ought each to make $x + 2$.

2dly, But if the second part when diminished by 2 be $x + 2$, then before it was diminished it was $x + 4$.

3dly, If the third part when doubled or multiplied by 2 be $x + 2$, then before it was multiplied, it must be the half of $x + 2$, that is, $\frac{x}{2} + 1$.

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Lastly,

Lastly, If half the fourth number be $x+2$, the fourth number itself must be $2x+4$; so that at last, the parts are found to be x , $x+4$, $\frac{x}{2}+1$, and $2x+4$; add these parts all together, and the sum will be $4x+\frac{x}{2}+9$; whence we have this equation, $4x+\frac{x}{2}+9=90$; therefore $8x+x+18=180$; therefore $9x=162$; and x , the 1st part $=18$; whence $x+4$, the 2d part $=22$; and $\frac{x}{2}+1$, the 3d part $=10$; and $2x+4$, the 4th part $=40$, as above.

PROBLEM 36.

61. *A shepherd driving a flock of sheep in time of war, meets a company of soldiers who plunder him of half his flock, and half a sheep over; the same treatment he meets with from a second, a third and a fourth company, every succeeding company plundering him of half the flock the last had left, and half a sheep over, insomuch that at last he had but 7 sheep left: I demand how many he had at first.*

Ans. His flock at first consisted of 127 sheep; and if the first company had only robbed him of half his flock, they would have left him $63\frac{1}{2}$ sheep; but as they plundered him of half his flock, and half a sheep over, they left him only 63 sheep; in like manner the second company left him 31, the third 15, and the fourth 7.

N. B. Before I enter upon the solution of this problem, I must put the learner in mind of what he has been told before, (introduction art. 13.) to wit, that a fraction may be halved two ways, either by halving the numerator, or doubling the denominator.

SOLUTION.

Put x for the number of his first flock; then had the first company only taken half his flock, they would have left him the other half, *viz.*

$\frac{x}{2}$; but they took half his flock and half a sheep over; therefore they

left him just so much less, to wit, $\frac{x}{2} - \frac{1}{2}$, or $\frac{x-1}{2}$: again, had the

second company only taken half what remained, they would have left him

half, to wit, $\frac{x-1}{4}$; but by taking half a sheep more, they left him

$\frac{x-1}{4} - \frac{1}{2}$, that is, $\frac{2x-2-4}{8}$, or $\frac{2x-6}{8}$, or $\frac{x-3}{4}$; in like man-

ner

Art. 61, 62, 63. PRODUCING SIMPLE EQUATIONS. 131

ner the third company left $\frac{x-3}{8} - \frac{1}{2}$, or $\frac{2x-6-8}{16}$, or $\frac{2x-14}{16}$, or $\frac{x-7}{8}$; and the last company left him $\frac{x-7}{16} - \frac{1}{2}$, or $\frac{x-15}{16}$; but they left him 7 sheep, by the supposition; therefore $\frac{x-15}{16} = 7$; and $x - 15 = 112$; and x his first number = 127, as above.

PROBLEM 37.

62. One buys a certain number of eggs, half whereof he buys in at 2 a penny, and the other half at three a penny; these he afterwards sold out again at the rate of 5 for twopence, and contrary to his expectation, lost a penny by the bargain: what was the number of his eggs?

Ans. The number of his eggs was 60; half whereof at two a penny cost him 15 pence; and the other half at three a penny, ten pence; and the whole 25 pence: but 60 eggs sold out at 5 for two pence, would only bring him in 24 pence, as appears by the rule of proportion; therefore he lost a penny by the bargain.

SOLUTION.

Put x for the number of eggs; then say, if 2 eggs cost one penny, what will $\frac{x}{2}$ one half of his eggs cost? and the answer will be $\frac{x}{4}$; and for the same reason the other half at three a penny will cost him $\frac{x}{6}$; so that for the whole he must pay $\frac{x}{4} + \frac{x}{6}$, or $\frac{5x}{12}$: again say, if 5 eggs were sold for two pence, what were x eggs sold for? and the answer will be $\frac{2x}{5}$; therefore $\frac{2x}{5}$ will be the number of pence he received for his eggs; subtract this from $\frac{5x}{12}$, the pence he paid for them, and the remainder $\frac{5x}{12} - \frac{2x}{5}$, or $\frac{x}{60}$ will be his loss; but by the supposition, he lost one penny; therefore $\frac{x}{60} = 1$; and x the number of eggs will be 60, as above.

PROBLEM 38.

63. One draws a certain quantity of wine out of a full vessel that held 81 gallons; and then recruiting the vessel with water, takes a second draught

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draught of as much wine and water together, as before he did of wine; and so he goes on for four draughts one after another, always taking the same quantity at a draught, and then recruiting the vessel with water; insomuch that at last, there were not above 16 gallons of pure wine left in the vessel, all the rest being water: I demand how much he took at every draught.

Ans. He took 27 gallons at every draught; therefore as there were 81 gallons of liquor in the vessel before every draught was made, he took $\frac{27}{81}$ or $\frac{1}{3}$ part of all the liquor in the vessel at every draught: but a mixture of wine and water, whenever it is made, may be supposed to be uniform; therefore in taking a third part of all the liquor in the vessel, he must not only take a third part of all the pure water in the vessel, but also a third part of all the pure wine; with this difference however, that the water was again recruited by filling up the vessel, whereas the wine was not: since then $\frac{1}{3}$ part of the wine was taken at every draught, it follows, that whatever quantity of pure wine was left in the vessel before any draught was made, there would only remain $\frac{2}{3}$ of that quantity afterwards; but the quantity of wine at the beginning was 81 gallons, *ex hypothesi*; therefore $\frac{2}{3}$ of 81, or 54 gallons must be left after the first draught; $\frac{2}{3}$ of 54, or 36 gallons after the second draught; $\frac{2}{3}$ of 36, or 24 after the third draught; and $\frac{2}{3}$ of 24, or 16 after the fourth draught, as the problem requires.

SOLUTION.

As there was the same quantity of liquor in the vessel before every draught was made, to wit, 81 gallons, and as there was taken the same quantity at every draught, it follows, that the same quantity must be left after every draught; for this put x ; then will $\frac{x}{81}$ shew what part or parts of all the liquor in the vessel was left after every draught, and consequently of the wine; whence the quantity of wine left in the vessel after every draught may be found thus; there were 81 gallons at the beginning, therefore $81 \times \frac{x}{81}$, or x , must be left after the first draught; $x \times \frac{x}{81}$, or $\frac{x^2}{81}$ must be left after the second draught; $\frac{x^2}{81} \times \frac{x}{81}$, or $\frac{x^3}{6561}$ after the third; and $\frac{x^3}{6561} \times \frac{x}{81}$, or $\frac{x^4}{531441}$ after the fourth; but according to the problem, there were left after the fourth draught 16 gallons; therefore $\frac{x^4}{531441} = 16$; therefore $x^4 = 8503056$; therefore $x^2 =$

$$\sqrt{8503056}$$

Art. 63, 64, 65. PRODUCING SIMPLE EQUATIONS. 133

$\sqrt{8503056} = 2916$; therefore x , or the number of gallons left in the vessel after every draught $= \sqrt{2916} = 54$; therefore $81 - x$, or the quantity taken at every draught must be 27, as above.

PROBLEM 39.

64. *It is required to divide the number 90 into two such parts, that one part may be to the other as 2 to 3.*

Ans. The numbers are 36 and 54: for in the first place, $36 + 54 = 90$; and in the next place, if both 36 and 54 be divided by 18, the quotients will be 2 and 3; whence I infer, that 36 is to 54 as 2 to 3; for a common division by the same number cannot alter the proportion of the numbers divided; and therefore if, after this common division, the quotients be to one another as 2 to 3, the dividends must be also in the same proportion.

SOLUTION.

Put x for the less part, and $90 - x$ for the other; then will x be to $90 - x$ as 2 to 3, by the supposition; but by art. 15, whenever there are four proportionals, the product of the extremes will be equal to the product of the middle terms: here the extremes are x and 3, whose product is $3x$; and the middle terms are $90 - x$ and 2, whose product is $180 - 2x$; therefore $3x = 180 - 2x$; therefore $5x = 180$; and x , the less part $= 36$; and $90 - x$, the greater $= 54$, as above.

PROBLEM 40.

65. *Two weights, one of 5 pounds, the other of 7, are suspended at the extremities of a very small rod 36 inches long: I demand the point in the rod, where these two weights will be in æquilibrium; that is, I demand the distance of this point from either extremity of the rod.*

Ans. The distance of the point of æquilibrium from the greater weight was 15 inches, and consequently $36 - 15$, or 21 inches from the less; for if the weights had been equal, the point of æquilibrium would have been exactly in the middle of the rod; but as one weight is less than the other, it will be proportionably nearer the greater weight; and thus 15 is to 21 as $\frac{5}{7}$ is to $\frac{21}{3}$, that is, as 5 to 7.

SOLUTION.

Put x for the distance of the point of æquilibrium from the greater weight, and $36 - x$ for it's distance from the less; then will x be to $36 - x$ as 5 to 7; whence by multiplying extremes and means, according

ing to art. 15, we have this equation, $7x = 180 - 5x$; whence $12x = 180$; and x , the distance of the point of *æquilibrium* from the greater weight $= 15$, and $36 - x$, it's distance from the less $= 21$, as above.

PROBLEM 41.

66. *What number is that, which being severally added to 36 and 52, will make the former sum to the latter as 3 to 4?*

Ans. The number is 12: for $36 + 12$ is to $52 + 12$, as 48 is to 64, as $\frac{48}{16}$ is to $\frac{64}{16}$, as 3 to 4.

SOLUTION,

Put x for the number sought, and you will have this proportion; $36 + x$ is to $52 + x$ as 3 to 4. Whence by multiplying extremes and means you will have $144 + 4x = 156 + 3x$; therefore $144 + x = 156$; therefore x , the number sought $= 12$, as above.

PROBLEM 42.

67. *A bookbinder sells me two paper books, one containing 48 sheets for 3 shillings and 4 pence, and another containing 75 sheets for 4 shillings and 10 pence, both bound at the same price, and both of the same sort of paper: I demand what he allows himself for binding.*

Ans. He reckoned 8 pence for binding; so that the price of the paper of the first book was 32 pence, and the price of the paper of the latter 50 pence: now if this answer be just, the two prices ought to bear the same proportion to one another as the two quantities of paper; and so we shall find them: for 32 pence are to 50 pence as $\frac{16}{2}$ are to $\frac{25}{2}$, that is, as 16 to 25; and 48 sheets are to 75 sheets as $\frac{48}{3}$ are to $\frac{75}{3}$, that is also, as 16 to 25.

SOLUTION,

Put x for the number of pence reckoned for binding; then we shall have $40 - x$ for the price of the paper in the first book, and $58 - x$ for the price of the paper in the second book; and $40 - x$ will be to $58 - x$ as 48 to 75; multiply extremes and means, and you will have this equation, $2784 - 48x = 3000 - 75x$; therefore $2784 + 27x = 3000$; therefore $27x = 216$; and x , the number of pence reckoned for binding $= 8$, as above.

PROBLEM 43.

68. *What number is that, which being severally added to 15, 27, and 45, will give three numbers in continual proportion?*

N. B.

Art. 68, 69. PRODUCING SIMPLE EQUATIONS. 135

N. B. Three numbers are said to be in continual proportion, when the first is to the second as the second is to the third.

Ans. The number sought is 9: for $15+9=24$; and $27+9=36$; and $45+9=54$; and 24 is to 36 as 36 is to 54: for 24 is to 36 as $\frac{24}{12}$ is to $\frac{36}{12}$, that is, as 2 to 3; and 36 is to 54 as $\frac{36}{18}$ is to $\frac{54}{18}$, that is also, as 2 to 3.

SOLUTION.

Put x for the number sought; then we shall have this proportion, $x+15$ is to $x+27$, as $x+27$ is to $x+45$; where the two middle terms are $x+27$ and $x+27$: multiply extremes and means, and you will have this equation, $xx+60x+675=xx+54x+729$; therefore $60x+675=54x+729$; therefore $6x+675=729$; therefore $6x=54$; and x , the number sought $=9$, as above.

PROBLEM 44.

69. One places a certain number of rods upright in a streight line, at equal distances one from another, the vacancies being no more than sufficient to contain two rods apiece; but finding that by this means, his line would not reach above 125 inches, he extended it to 208 inches by opening the vacancies just as wide again as before: What was his number of rods?

Ans. The number of rods was 84, and consequently the number of intervals 83: for if two rods admit but of one interval, three rods of 2, &c, 84 rods will admit of 83 intervals, which intervals if they were to be filled, would take up 83×2 , or 166 rods; therefore if the first line had been full, it would have taken up $84+166$, or 250 rods: again, the number of rods sufficient to fill the vacancies of the second line was 83×4 , or 332; therefore if the second line had been full, it would have taken up $84+332$, or 416 rods: now if this answer be just, the lengths of these two lines ought to have the same proportion to one another, as have the number of rods they would have taken up had they been full; and so we shall find them: for 125 inches are to 208 inches as 125×2 to 208×2 , that is, as 250 rods to 416 rods, as was to be demonstrated.

SOLUTION.

Put x for the number of rods; then will the number of vacancies be $x-1$; and the number of rods sufficient to fill the vacancies of the first line, $2x-2$; and the number of rods the first line would have taken, had it been full, $3x-2$: again, the number of rods sufficient to fill the vacancies of the second line will be $4x-4$; and therefore the number of

of rods the second line would have taken, had it been full, will be $5x-4$; whence we shall have this proportion, $3x-2$ is to $5x-4$ as 125 is to 208; and by multiplying extremes and means, this equation, $624x-416=625x-500$; therefore $x-500=-416$; therefore x , the number of rods $=84$, as above.

Of the method of resolving problems wherein more unknown quantities than one are concerned, and represented by different letters.

70. Hitherto we have used but one single letter in every problem for some one unknown quantity in it; and if there were more, the rest received their names from the conditions of the problem: but in cases of a more complicated nature, where many unknown quantities are linked and entangled in one another, this method will be found very difficult; and therefore in such cases, the Algebraist is allowed to use as many different letters as he has unknown quantities, provided he finds out as many independent equations for discovering their values; see art. 92: for though in every equation wherein more unknown quantities than one are concerned, they hinder one another from being found out, yet if as many fundamental equations at first be given, as there are unknown quantities, it will not be difficult in many cases, from these to derive others that are more simple, till at last you come to an equation wherein but one only unknown quantity is concerned, in which case all the rest are said to be exterminated.

Whenever two or more equations are proposed, involving as many unknown quantities, these equations must first be prepared by freeing them from fractions wherever there are any, and by ordering every particular equation so, that all the unknown quantities may possess one side of the equation, and such as are known the other; or else, that all the quantities may possess one side of the equation, and a cypher the other: it will be also convenient, that in every particular equation, the unknown quantities be placed in the same order.

In laying down rules for exterminating unknown quantities, I shall begin with the simplest case first, which is that of two equations and two unknown quantities; and when I have given as many examples as shall be thought proper in this case, I shall then proceed to others where more unknown quantities are to be exterminated.

But here I must not forget to advertise the reader, that as I am now treating of simple equations, and problems producing such equations, I shall not meddle with any cases of extermination which lead to equations
of

Art. 70. PRODUCING SIMPLE EQUATIONS. 137

of higher forms: when I come to treat of quadratic equations, I may then perhaps add something further upon this subject; but to undertake to explain all the various methods of exterminating unknown quantities would be an endless task, and a most intolerably laborious and tedious one both to the writer and the reader, whom I cannot yet suppose to be so far gone in Analytics, as to be willing to purchase this sort of knowledge at any rate.

Let then x and y be two unknown quantities to be found out by the help of the two following equations, $4x - 5y = 2$, and $6x - 7y = 4$; or the question may be stated thus; if $4x - 5y = 2$, and $6x - 7y = 4$, what are x and y ? now as these equations want no preparation, put them down one under another; then upon a bye piece of paper multiply the first equation ($4x - 5y = 2$) by 6 the coefficient of x in the second equation, and the product will give this equation, $24x - 30y = 12$; again, multiply the second equation ($6x - 7y = 4$) by 4, the coefficient of x in the first equation, and the product gives $24x - 28y = 16$; subtract now either of these two last equations from the other, and x will be exterminated: I choose in the present case to subtract the former equation from the latter, that the coefficient of y after subtraction may affirmative, thus;

$$\begin{array}{r} 24x - 28y = 16 \\ 24x - 30y = 12 \\ \hline * + 2y = 4. \end{array}$$

From this subtraction you have the following equation, $2y = 4$, which put down under the two first equations to make a third; then resolve this third equation $2y = 4$, and you will have $y = 2$, which put down under the rest for a fourth equation.

Having thus found the value of $y = 2$, put this value instead of y in the more simple of the two first equations, suppose in the equation $4x - 5y = 2$, and you will have $4x - 10 = 2$; whence $4x = 12$, and $x = 3$, which put down for a fifth equation, and the work is done; for x is now found equal to 3, and y equal to 2, and these numbers three and two being substituted for x and y respectively, will answer both the conditions of the question, that is, you will have $4x - 5y = 12 - 10 = 2$, and $6x - 7y = 18 - 14 = 4$.

$$\begin{array}{ll} \text{1st Equ.} & 4x - 5y = 2. \\ \text{2d,} & 6x - 7y = 4. \\ \text{3d,} & * + 2y = 4. \\ \text{4th,} & * \quad y = 2. \\ \text{5th,} & x \quad * = 3. \end{array}$$

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The coefficients of x , the quantity to be exterminated in the two first equations were 4 and 6: now as these numbers admit of a common divisor without any remainder, namely 2, divide them both by 2, and the quotients will be 2 and 3; use now these numbers 2 and 3 instead of 4 and 6, and the operation as well as the equation resulting from it, will become more simple: for the first equation multiplied by 3 instead of 6, gives $12x - 15y = 6$; and the second equation multiplied by 2 instead of 4, gives $12x - 14y = 8$; and the difference of these two equations is $y = 2$.

Another way of exterminating the unknown quantity x , is as follows: find out the value of x in respect of y , in the more simple of the two first equations; then substitute this value instead of x in the other equation, you will have an equation, wherein y alone is concerned: thus in the foregoing example, the first equation was $4x - 5y = 2$, therefore $4x = 5y + 2$, and $x = \frac{5y+2}{4}$; substitute now this value $\left(\frac{5y+2}{4}\right)$ instead of x in the second equation, $6x - 7y = 4$, by making $6x = \frac{30y+12}{4}$, and you will have this equation, $\frac{30y+12}{4} - 7y = 4$; therefore $30y + 12 - 28y = 16$; therefore $2y + 12 = 16$; whence $2y = 4$, and $y = 2$; and x , or $\frac{5y+2}{4} = 3$, as before.

N. B. 1st, What has here been said concerning the extermination of the quantity x , may as well be applied to the other quantity y , except that it's coefficients 5 and 7 will not admit of a common divisor, as did the numbers 4 and 6.

2^{dly}, Of the two different ways of extermination here laid down, sometimes one will be found more expeditious, and sometimes the other, as will appear by the following problems.

3^{dly}, In the case of two unknown quantities, if the value of either of them can be had in integral terms in both equations, equate the two values one to the other, and you will have the other unknown quantity, by means whereof the first will also be known; and this makes a third way of extermination, whereof there are so many examples in the following problems, that nothing more needs here to be said of it.

Whenever two quantities, as x and y , are multiplied together to produce a third xy , the two multiplicands x and y are called factors, or efficient, in which case, each is said to be the others coefficient: thus in the quantity xy , x is said to be the coefficient of y , and y the coefficient of x ; therefore if in any quantity wherein x is concerned as an efficient,

Art. 70, 71. PRODUCING SIMPLE EQUATIONS. 139

ent, it's coefficient be desired; divide that quantity by x , and the quotient will be the coefficient: thus if the quantity $12x - yx$ be divided by x , the quotient is $12 - y$; therefore in the quantity $12x - yx$, the coefficient of x is $12 - y$

A D V E R T I S E M E N T.

The reader must now no longer expect to have all simple equations resolved to his hand, as hitherto has been done. If after sixteen examples of simple equations resolved, and the solution of forty four Algebraic problems, he be still at a loss how to reduce a simple equation, it must proceed from a weakness that either admits of no cure, or deserves none.

P R O B L E M 45.

71. *What two numbers are those, the product of whose multiplication is 144, and the quotient of the greater divided by the less is 16?*

S O L U T I O N.

Put x for the greater number, and y for the less; and the question when abstracted from words will stand thus: if $xy = 144$, and $\frac{x}{y} = 16$, what are x and y ?

The first of these equations wants no preparation, and therefore may be put down thus;

$$\text{Equ. 1st, } xy = 144.$$

The second equation, when prepared according to the last art. will stand thus; Equ. 2d, $x - 16y = 0$.

Multiply the first equation by 1, the supposed coefficient of x in the second, and the equation not being altered by such a multiplication, will be $xy = 144$; multiply also the second equation by y , which according to the foregoing art. is the coefficient of x in the first, and you will have $xy - 16yy = 0$; subtract this latter product from the former, and you will have,

$$\text{Equ. 3d, } * 16yy = 144; \text{ whence}$$

$$\text{Equ. 4th, } * y = 3.$$

Substitute now 3 instead of y , or $3x$ instead of xy in the first equation, and you will have $3x = 144$, and consequently,

$$\text{Equ. 5th, } x = 48.$$

So that the numbers at last are found to be 48 and 3; and they will answer the conditions of the question: for $48 \times 3 = 144$, and $\frac{48}{3} = 16$.

$$\begin{aligned}
 \text{Equ. 1st, } xy &= 144. \\
 2\text{d, } x - 16y &= 0. \\
 3\text{d, } * \quad 16yy &= 144. \\
 4\text{th, } * \quad y &= 3. \\
 5\text{th, } x &= 48.
 \end{aligned}$$

Another solution of the foregoing problem, from the last article.

Having found from the second equation that $x = 16y$, put $16y$ for x , or $16yy$ for xy in the first equation, and you will have $16yy = 144$; whence y and x may be found as before.

PROBLEM 46.

72. It is required to find two numbers with the following properties, to wit, that the first with half the second may make 20; and moreover, that the second with a third part of the first may make 20.

SOLUTION.

Put x for the first number, and y for the second, and the fundamental equations will be $x + \frac{y}{2} = 20$, and $y + \frac{x}{3} = 20$; which being prepared according to art. 70, will stand thus;

$$\text{Equ. 1st, } 2x + y = 40.$$

$$\text{Equ. 2d, } x + 3y = 60.$$

Subtract the first equation from twice the second, and you will have

$$\text{Equ. 3d, } * \quad 5y = 80; \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 16.$$

Put 16 instead of y in the first equation, and you will have $2x + 16 = 40$, whence

$$\text{Equ. 5th, } x = 12.$$

Therefore the numbers sought are 12 and 16, and not 16 and 12, though 16 was found first; because $x = 12$ was put for the first number. That these numbers will answer the conditions of the question is plain: for $12 + \frac{16}{2}$ or $12 + 8 = 20$; and $16 + \frac{12}{3}$, or $16 + 4 = 20$.

Another solution from art. 70.

Having found from the second equation that $x = 60 - 3y$, put $60 - 3y$ for x , or $120 - 6y$ for $2x$ in the first equation, and you will have $120 - 6y + y = 40$; whence $y = 16$, as before.

PROB-

Art. 73, 74. PRODUCING SIMPLE EQUATIONS. 141

PROBLEM 47.

73. One exchanges 6 french crowns and two french dollars for 45 shillings; and at another time 9 crowns and 5 dollars of the same coin for 76 shillings: I demand the distinct values of a crown and of a dollar.

SOLUTION.

Put x and y for the number of shillings a crown and a dollar are respectively worth, and the equations will stand thus;

$$\text{Equ. 1st, } 6x + 2y = 45.$$

$$\text{Equ. 2d, } 9x + 5y = 76.$$

Subtract 3 times the first equation from twice the second, and you will have

$$\text{Equ. 3d, } * \quad 4y = 17; \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 4\frac{1}{4} \text{ shillings;}$$

that is, 4 shillings and 3 pence; put now $4\frac{1}{4}$ for y , or $8\frac{1}{2}$ for $2y$ in the first equation, and you will have $6x + 8\frac{1}{2} = 45$, and $6x = 36\frac{1}{2}$, and

$$\text{Equ. 5th, } x = 6\frac{1}{12};$$

that is, $6\frac{1}{12}$ shillings, or 6 shillings and a penny; therefore the value of a crown was 6 shillings and a penny, and that of a dollar 4 shillings and 3 pence; and these values will answer the conditions of the question; for at this rate, 6 crowns will amount to 36 shillings and 6 pence, 2 dollars to 8 shillings and 6 pence, and the whole to 45 shillings; moreover, 9 crowns will amount to 54 shillings and 9 pence, 5 dollars to 21 shillings and 3 pence, and the whole sum to 76 shillings.

PROBLEM 48.

74. It is required to find two such numbers, that half the first together with a third part of the second may make 32; and moreover, that a fourth part of the first together with a fifth part of the second may make 18.

SOLUTION.

Put x and y for the two numbers, and the fundamental equations will be $\frac{x}{2} + \frac{y}{3} = 32$, and $\frac{x}{4} + \frac{y}{5} = 18$; which equations when duly prepared, will stand thus;

$$\text{Equ. 1st, } 3x + 2y = 192.$$

$$\text{Equ. 2d, } 5x + 4y = 360.$$

Subtract 5 times the first equation from 3 times the second, and you will have

have

Equ. 3d, * $2y=120$; whenceEqu. 4th, * $y=60$;whence, and from the first equation, you will have $3x+2y$, or $3x+120=192$, which givesEqu. 5th, $x=24$.

So the numbers are 24 and 60; and they will answer the conditions of the question: for $\frac{24}{2}+\frac{60}{3}$, that is, $12+20=32$; and moreover, $\frac{24}{4}+\frac{60}{5}$, that is, $6+12=18$.

PROBLEM 49.

75. Two persons A and B were talking of their ages; says A to B, 7 years agoe I was just three times as old as you were, and 7 years hence I shall be just twice as old as you will be: I demand their present ages.

SOLUTION.

Let a and b represent the present ages of A and B respectively; then their ages 7 years agoe were $a-7$ and $b-7$, and their ages 7 years hence will be $a+7$ and $b+7$; whence, and from the conditions of the problem, may be derived the two following fundamental equations;

$$a-7=b-7 \times 3 = 3b-21, \text{ and}$$

$$a+7=b+7 \times 2 = 2b+14.$$

From the former of these two equations, to wit, $a-7=3b-21$, we have $a=3b-14$; from the second equation, to wit, $a+7=2b+14$, we have $a=2b+7$; therefore $3b-14=2b+7$, since both are equal to a ; whence $b=21$, and $2b+7$, or $a=49$.

A therefore was 49 years old, and B 21 years old; which is true: for then, 7 years before, A's age would be 42, and B's 14; and 42 is three times 14: on the other hand, 7 years after, A's age would be 56, and B's 28; and 56 is twice 28.

PROBLEM 50.

76. A jocky has two horses, A and B, whose values are sought; he has also two saddles, one valued at 12 pounds, the other at 2: now if he sets the better saddle upon A and the worse saddle upon B, A will then be worth twice as much as B; but on the other hand, if he sets the better saddle upon B, and the worse saddle upon A, B will then be worth three times as much as A: I demand the values of the horses.

SOLUTION.

Let a and b represent the prices of the two horses A and B respectively in pounds; then if the better saddle be set upon A, and the worse upon

Art. 76, 77, 78. PRODUCING SIMPLE EQUATIONS. 143
 on B , A will be worth $a+12$, and B will be worth $b+2$, and the first fundamental equation will be $a+12=\overline{b+2} \times 2=2b+4$; on the other hand, if the better saddle be set upon B , and the worse upon A , then B will be worth $b+12$, and A will be worth $a+2$, and the second fundamental equation will be $b+12=\overline{a+2} \times 3=3a+6$: in the first fundamental equation, where $a+12=2b+4$, we have $a=2b-8$; substitute therefore $2b-8$ instead of a , or rather $6b-24$ instead of $3a$, in the second fundamental equation, (which is $3a+6=b+12$) and you will have $6b-24+6$, that is, $6b-18=b+12$; whence $b=6$, and $2b-8$, or $a=4$: A then was valued at 4 pounds, and B at 6, and they will answer the conditions of the question, as any one may easily try.

PROBLEM 51.

77. *There is a certain fraction, which if an unit be added to the numerator, will be equal to $\frac{1}{3}$; but if on the contrary an unit be added to the denominator, the fraction will then be equivalent to $\frac{1}{4}$: I demand the numerator and denominator of the fraction.*

SOLUTION.

Call the fraction $\frac{x}{y}$, and you will have these two fundamental equations, $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$: the former of these equations when reduced, gives $y=3x+3$, and the latter gives $y=4x-1$; therefore $4x-1=3x+3$, because both are equal to y ; whence x the numerator of the fraction is 4; and $3x+3$, or y , the denominator is 15; and the fraction itself is, $\frac{4}{15}$; which if an unit be added to the numerator, will be $\frac{5}{15}$, or $\frac{1}{3}$; but if an unit be added to the denominator, it will be $\frac{4}{16}$, or $\frac{1}{4}$.

PROBLEM 52.

78. *There is a certain fishing rod consisting of two parts, whereof the upper part is to the lower as 5 to 7; and moreover 9 times the upper part together with 13 times the lower, is equal to 11 times the whole rod and 36 inches over: I demand the length of the two parts.*

SOLUTION.

Put x for the length of the upper part in inches, and y for the lower; then will $x+y$ be the length of the whole rod; and since x is to y as 5 to 7 *ex hypothesi*, by multiplying extremes and means according to art.

15, you will have $7x = 5y$ for a fundamental equation: again, as 9 times the upper part, together with 13 times the lower, is equal to 11 times the whole rod, and 36 inches over, you have $9x + 13y = 11x + 11y + 36$ for a second fundamental equation: the latter of these two equations gives $x = y - 18$, and consequently $7x = 7y - 126$; substitute this value instead of $7x$, in the first fundamental equation, where $7x = 5y$, and you will have $7y - 126 = 5y$; whence $y = 63$; and $y - 18$, or $x = 45$.

The upper part therefore was 45 inches, and the lower 63, as will appear upon tryal.

PROBLEM 53.

79. One lays out 2 shillings and sixpence in apples and pears, buying his apples at four, and his pears at five a penny; and afterwards accommodates his neighbour with half his apples and one third part of his pears for thirteenpence, which was the price he bought them at: I demand how many he bought of each sort.

SOLUTION.

Put x for the number of apples, and y for the number of pears; then if 4 apples cost one penny, x will cost $\frac{x}{4}$ pence; and for the same reason y will cost $\frac{y}{5}$ pence, and you will have $\frac{x}{4} + \frac{y}{5} = 30$ for a first fundamental equation: again, the price of $\frac{x}{2}$, half of his apples will be $\frac{x}{8}$; and the price of $\frac{y}{3}$, a third part of his pears will be $\frac{y}{15}$; and you will have $\frac{x}{8} + \frac{y}{15} = 13$ for a second fundamental equation. Hence

$$\text{Equ. 1st, } 5x + 4y = 600.$$

$$\text{Equ. 2d, } 15x + 8y = 1560.$$

Subtract the second equation from three times the first, according to art. 70, and you will have

$$\text{Equ. 3d, } * \quad 4y = 240; \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 60.$$

Substitute now 60 for y , that is, 240 for $4y$ in the first equation $5x + 4y = 600$, and you will have $5x + 240 = 600$; whence

$$\text{Equ. 5th, } x \quad * = 72.$$

Therefore the number of apples was 72, and the number of pears 60, as will appear upon tryal.

PROB-

Art. 80, 81. PRODUCING SIMPLE EQUATIONS. 145

PROBLEM 54.

80. *A greyhound spying a hare at the distance of fifty of his own leaps from him, pursues her with full speed, making three leaps for every four of the hare's; and moreover passing over as much ground in two leaps, as the hare did in three: I demand how many leaps each made during the whole course.*

SOLUTION.

The number of the dog's leaps during the whole course, x ;

The number of the hare's leaps in the same time, y ;

Therefore while the dog makes x leaps, the hare makes y ; but according to the problem, while the dog made three leaps, the hare made four; therefore x is to y as 3 to 4; whence by multiplying extremes and means we have $4x = 3y$: again, from the hare's form to the end of the course, the dog made $x - 50$ leaps, and passed over as much ground as the hare did in all her's y ; but according to the problem, the dog passed over as much ground in two leaps as the hare did in three; therefore $x - 50$ is to y as 2 to 3; whence again by multiplying extremes and means we have $3x - 150 = 2y$: the rest of the solution is as follows;

$$\text{Equ. 1st, } 4x - 3y = 0.$$

$$\text{Equ. 2d, } 3x - 2y = 150.$$

Subtract three times the first equation from four times the second, and you will have

$$\text{Equ. 3d, } * \quad y = 600.$$

Put 600 for y in the first equation, and you will have $4x - 3y$, that is, $4x - 1800 = 0$; whence

$$\text{Equ. 4th, } x \quad * = 450.$$

Therefore the dog made 450 leaps, and the hare 600 during the whole course; and 450 is to 600 as $\frac{450}{150}$ is to $\frac{600}{150}$, that is, as 3 to 4: again, from the hare's form to the end of the course the dog made 400 leaps; and 400 is to 600 as 4 to 6, or as 2 to 3.

PROBLEM 55.

81. *It is required to find two numbers x and y of such a nature, that if both be multiplied by 18, the first product will be a square, and the second will be the side or root of that square; but if both be multiplied by 3, the first product will be a cube, and the second the side of that cube.*

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SOLUTION.

If x and y be both multiplied by 18, the products will be $18x$ and $18y$, whereof the former is to be a square, and the latter the side of that square, that is, in other words, the former product is to be equal to the square of the latter; and so we have $18x = 324yy$, and $x = 18yy$: again, if x and y be multiplied by 3, the products will be $3x$ and $3y$, whereof the former is to be equal to the cube of the latter; and so we have $3x = 27y^3$, and $x = 9y^3$; therefore $9y^3 = 18y^2$, because both are found equal to x ; therefore $y = 2$, and $9y^3$, or $x = 72$; so that the numbers sought are 72 and 2, which will answer the conditions; for if both be multiplied by 18, the products will be 1296 and 36, whereof the former is a square, and the latter it's side; but if the numbers 72 and 2, be both multiplied by 3, the products will be 216 and 6, whereof the former is a cube, and the latter it's side.

PROBLEM 56.

82. *There is a certain floor in form of a rectangle or long square, whose dimensions are such, that if it had been two feet broader, and three feet longer, it would have been sixty four square feet larger; but if on the other hand, it had been three feet broader and two feet longer, it would then have been sixty eight square feet larger: I demand the length and breadth of the floor.*

SOLUTION.

Put x for the breadth, and y for the length of the floor; then will it's area, or number of square feet it contains be $x \times y$ or xy , as every one knows; but if it had been 2 feet broader and 3 feet longer, it's area would then have been $x + 2 \times y + 3$, that is, $xy + 3x + 2y + 6$; and as by supposition, this area is to exceed the true one by 64 square feet, we have the following equation, $xy + 64 = xy + 3x + 2y + 6$: again, if this floor had been 3 feet broader and 2 feet longer, its area would then have been $x + 3 \times y + 2$, that is, $xy + 2x + 3y + 6$; and this last area is to exceed the true one by 68 square feet; whence we have this equation, $xy + 68 = xy + 2x + 3y + 6$; whence

$$\text{Equ. 1st, } 3x + 2y = 58.$$

$$\text{Equ. 2d, } 2x + 3y = 62.$$

Subtract twice the first equation from three times the second, and you will have

$$\text{Equ. 3d, } * \quad 5y = 70; \text{ and}$$

$$\text{Equ. 4th, } * \quad y = 14.$$

Sub-

Art. 82, 83. PRODUCING SIMPLE EQUATIONS. 147

Substitute now 14 for y , or 28 for $2y$ in the first equation $3x+2y=58$, and you will have $3x+28=58$, whence

$$\text{Equ. 5th, } x = 10.$$

The floor therefore was 10 feet broad, and 14 feet long; and its area 140 square feet, which answers the conditions of the question: for then, if it had been 2 feet broader and 3 feet longer, its area would have been $12 \times 17 = 204 = 140 + 64$; but if it had been 3 feet broader and 2 feet longer, its area would have been $13 \times 16 = 208 = 140 + 68$.

PROBLEM 57.

83. *A certain company at a tavern found, when they came to pay their reckoning, that if they had been three more in company to the same reckoning, they might have paid one shilling apiece less than they did; and that, had they been two fewer in company, they must have paid one shilling apiece more than they did: I demand the number of persons, and their quota.*

SOLUTION.

Put x for the number of persons, and y for the number of shillings every one actually paid; now if 4 persons are to pay 5 shillings apiece, the whole reckoning must be 4×5 or 20 shillings; therefore if x persons are to pay y shillings apiece, the whole reckoning must be $y \times x$ or xy shillings: this being laid down, suppose them now to be three more in company; then will the number of persons be $x+3$; and to find what every particular person ought to pay in this case, the whole reckoning xy , must be divided by $x+3$, the number of persons, and the quotient

$\frac{xy}{x+3}$ will be every one's particular reckoning; but according to the problem, every one's particular reckoning in this case would have been

one shilling less than it actually was, that is, $y-1$; therefore $\frac{xy}{x+3} = y-1$; in like manner the second condition of the problem furnishes this

equation, $\frac{xy}{x-2} = y+1$: the first of these equations, to wit, $\frac{xy}{x+3} = y-1$, being reduced, gives $x=3y-3$; and the second equation, to wit,

$\frac{xy}{x-2} = y+1$ being reduced gives $x=2y+2$; therefore $3y-3=2y+2$, and $y=5$; whence $2y+2$, or $x=12$.

So there were 12 persons in company, their reckoning 5 shillings apiece, and their whole reckoning 3 pounds, or 60 shillings; which answers the conditions of the question: for $\frac{60}{12}=5$, and $\frac{60}{10}=6$.

PROBLEM 58.

84. *A vintner has two sorts of wines, a better sort and a worse, whose values are such, that if he mixes them in the proportion of two to three, that is, after the rate of two quarts of the better sort to three of the worse, the mixture will be worth one and twenty pence a quart; but if he mixes them in the proportion of seven to eight, that is, after the rate of seven quarts of the better sort to eight of the worse, the mixture will then be worth two and twenty pence the quart: I demand the price of a quart of each sort.*

SOLUTION.

Put x for the price in pence of a quart of the better sort, and y for the price of a quart of the worse in pence: then if he mixes 2 quarts of the better sort with 3 of the worse, the mixture will be worth $2x + 3y$; but as there were 5 quarts of this mixture, valued at 21 pence a quart, the whole mixture must be worth 105 pence; whence we have this equation, $2x + 3y = 105$: in like manner the other condition of the problem furnishes this equation, $7x + 8y = 22 \times 15 = 330$; which equations are solved as follows;

Equ. 1st,	$2x + 3y = 105.$		Equ. 4th,	*	$y = 15.$
Equ. 2d,	$7x + 8y = 330.$		Equ. 5th,	x	$* = 30.$
Equ. 3d,	$* \quad 5y = 75.$				

So the better sort of wine was worth 30 pence a quart, and the worse 15; which answers the conditions of the problem: for at this rate, two quarts of the better sort will be worth 60 pence, three of the worse 45 pence, the whole five quarts 105 pence, and a single quart 21 pence; again, seven quarts of the better sort will be worth 210 pence, eight of the worse 120 pence, the whole fifteen quarts 330 pence, and a single quart 22 pence.

PROBLEM 59.

85. *There is a certain number consisting of two places, which is equal to four times the sum of it's digits; and if to the number be added 18, the digits will be inverted: I demand the number.*

N. B. By the digits of any number are meant the figures that represent it, without any regard had to their places; thus the digits of the number 36 are 3 and 6, whose sum is 9; whence it appears, that the number 36 has the first property described in the problem, to wit, that it is equal to 4 times 9, or 4 times the sum of it's digits; but it is not the true number sought, because it has not the second property; for if it had, then 36 and 18 together would make 54, a number with the former digits inverted; whereas they only make 54.

SOLU-

Art. 85, 86, 87. PRODUCING SIMPLE EQUATIONS. 149
SOLUTION.

Put x for the digit in the ten's place, and y for the digit in the place of units, and the sum of the digits will be $x + y$; but how to read such a number as this, that hath x in the ten's place and y in the place of units, comes next to be considered: now in order to do this, the reader must reflect what passes in his own mind when he sees a number of two places, as 36; then he will easily see, that he makes the 6 in the place of units stand for no more than it's value, but the 3 in the place of tens he makes to stand not for 3, but for 30, a number 10 times as much; so that he reads the number 36, *quasi* $30 + 6$; therefore a number that has x in the place of tens, and y in the place of units, ought to be read thus, $10x + y$; and a number with the same digits inverted must be read thus, $10y + x$: this being well understood, the fundamental equations flow easily from the conditions of the problem thus; $10x + y = 4x + 4y$, and $10x + y + 18 = 10y + x$. The former equation reduced gives $y = 2x$, and the latter gives $y = x + 2$; therefore $2x = x + 2$; whence $x = 2$, $2x$ or $y = 4$, and the number sought is 24; the sum of whose digits is 6: and that it answers the conditions of the problem, is plain; for $24 = 4$ times 6; and $24 + 18 = 42$.

A L E M M A.

86. *If there be a compound quantity consisting of any two multiples of x and y , whereof one is affirmative and the other negative, and if this compound quantity is to be subtracted from $x + y$; this subtraction may be easily made without the work of the pen, thus: change the signs of the two parts to be subtracted; then encrease the coefficient of the affirmative part, and diminish the coefficient of the negative part by unity, and you will have the remainder.* Thus if $6x - 10y$ is to be subtracted from $x + y$, first change the signs $6x - 10y$ and you will have $10y - 6x$; then encrease the affirmative coefficient 10, and diminish the negative coefficient 6 by unity, and you will have $11y - 5x$ for a remainder, as may easily be seen by actually subtracting $6x - 10y$ from $x + y$: again, if $2y$, that is, $2y - 0x$ is to be subtracted from $x + y$, the remainder will be $x - y$.

P R O B L E M 60.

87. *Two persons A and B have each a certain number of counters; A gives to B as many as B has already; after which, B returns back again to A as many as A has left; then A returns again to B as many as B has left, and so they go on; and after four exchanges thus made, they had each of them sixteen counters; I demand how many each had at first.*

S O L U -

SOLUTION.

Put x for A 's number of counters, and y for that of B 's; then it is plain that the sum of both their counters at the beginning will be $x+y$; and if so, then it is as plain, that the sum of both their counters will ever afterwards be $x+y$; because, whatever exchanges were made backwards and forwards, there were no counters lost between them: therefore if in any case an expression for either of their counters be given, it is but subtracting that expression from $x+y$, and the remainder will be an expression for the other's counters; whence we have the following solution;

A 's number of counters, x .

B 's, y .

B 's after the first exchange, $2y$.

A 's, $x-y$.

A 's after the second exchange, $2x-2y$.

B 's, $3y-x$.

B 's after the third exchange, $6y-2x$.

A 's, $3x-5y$.

A 's after the fourth exchange, $6x-10y$.

B 's, $11y-5x$.

And thus may the exchanges be carried on with ease to any number at pleasure: but according to the problem, after four exchanges thus made, they had each of them 16 counters; whence we have these two equations, $6x-10y=16$, and $11y-5x=16$: the resolution follows;

Equ. 1st,	$3x-5y=8$.	Equ. 4th,	*	$y=11$.
2d,	$5x-11y=-16$.	5th,	x	$*=21$.
3d,	* $8y=88$.			

So that A had 21 counters, and B 11; and the account will stand thus:
 After the first exchange, B has 22 counters, and A has $21-11=10$.
 After the second exchange, A has 20, and B has $22-10=12$.
 After the third exchange, B has 24, and A has $20-12=8$.
 After the fourth exchange, A has 16, and B has $24-8=16$.

Observations upon the foregoing problem.

If any one has curiosity enough to enquire by what contrivance in this and many other problems, such *data* can be made choice of as that the answers shall come out in whole numbers; whereas, were those *data* assumed arbitrarily and at random, the answers would, generally speaking, come out in fractional numbers; this is effected by the following artifice: suppose that in this problem, after four exchanges made between A and B ,

I had

Art. 87. PRODUCING SIMPLE EQUATIONS. 151

I had a mind they should both have the same number of counters, but did not know what number to assign them at last, so as that each should have a whole number of counters at first; I put down some letter, as c , for the number of counters each had left at last; then the two fundamental equations will stand thus, $6x - 10y = c$, and $11y - 5x = c$; and must be resolved thus:

$$\begin{array}{l|l} \text{Equ. 1st, } 6x - 10y = c. & \text{Equ. 3d, } * \quad 16y = 11c. \\ \text{Equ. 2d, } 5x - 11y = -c. & \text{Equ. 4th, } * \quad y = \frac{11c}{16}. \end{array}$$

Therefore by the first equation, $6x - 10y$, that is, $6x - \frac{110c}{16} = c$; therefore $96x - 110c = 16c$; therefore $96x = 126c$; therefore

$$\text{Equ. 5th, } x * = \frac{126c}{96} = \frac{21c}{16}.$$

So that, to make each person A and B , have the same number c of counters at last, A 's first number must be $\frac{21c}{16}$, and B 's $\frac{11c}{16}$; whence it follows, that if for the number c , I make choice of any number that can be divided by 16 without any remainder, then both A and B will necessarily have a whole number at first; for if $\frac{c}{16}$ be a whole number,

then both $\frac{21c}{16}$ and $\frac{11c}{16}$ must be so too; and this is the reason that I made choice of the number 16 itself to stand for c in this problem; for then I knew, not only that the answer would come out in whole numbers, but also, that this problem would be the simplest of its kind, since 16 is the least number that can be divided by 16.

This problem may also be solved by the help of one single letter only, thus: make c now to represent, not the number of counters each had at last, but the sum of both their counters at last, and consequently the sum of both their counters all along, and the lemma for carrying on the exchanges to any number at pleasure will be this; *If there be a compound quantity consisting of any two multiples of c and x , whereof one is affirmative and the other negative, and if this compound quantity is to be subtracted from c , the remainder may be obtained, first by changing the signs of the two parts, and then by encreasing or diminishing the coefficient of c by unity, according as it happens after this change to be affirmative or negative.* Thus if $2c - 2x$ is to be subtracted from c , the remainder will be $2x - c$; and if $4x - 2c$ is to be subtracted from c , the remainder will be $3c - 4x$; whence we have this solution of the foregoing problem:

A 's

A's number of counters, x .

B's, $c - x$.

B's after the first exchange, $2c - 2x$.

A's, $2x - c$.

A's after the second exchange, $4x - 2c$.

B's, $3c - 4x$.

B's after the third exchange, $6c - 8x$.

A's, $8x - 5c$.

A's after the fourth exchange, $16x - 10c$.

B's, $11c - 16x$.

But according to the problem, after the fourth exchange each had $\frac{1}{2}c$: therefore x may be found, and will be the same whether we make $16x$

$- 10c = \frac{c}{2}$, or $11c - 16x = \frac{c}{2}$: if we make $16x - 10c = \frac{c}{2}$, we

shall have $32x - 20c = c$, and $32x = 21c$, and x , or *A's* number

$= \frac{21c}{32}$; whence $c - x$, or *B's* number $= \frac{11c}{32}$; therefore the simplest

problem of this kind (I mean in the case of four exchanges) that will admit of a whole number solution, is when c is made equal to 32.

PROBLEM 61.

88. *What two numbers are those, whose sum is twice, and the product of whose multiplication is twelve times their difference?*

SOLUTION.

Put x for the greater number, and y for the less; then will their difference be $x - y$, their sum $x + y$, and the product of their multiplication xy or yx ; and the equations will be $x + y = 2x - 2y$, and $yx = 12x - 12y$; whence

Equ. 1st, $x - 3y = 0$.

Equ. 2d, $12x - yx - 12y = 0$.

Multiply the first equation by $12 - y$, which by art. 70, is the coefficient of x in the second, and the product will be $12x - yx - 36y + 3yy = 0$; subtract this equation from the second, and you will have

Equ. 3d, $24y - 3yy = 0$; whence

Equ. 4th, $y = 8$; and

Equ. 5th, $x = 24$.

And the numbers 24 and 8 will answer the conditions.

Otherwise thus: by the first equation $x = 3y$, and $4x = 12y$; substitute $4x$ for $12y$ in the second equation, and you will have $12x - yx - 4x = 0$; divide by x , and you will have $12 - y - 4 = 0$, and $y = 8$, and x or $3y = 24$, as before.

PROB-

PROBLEM 62.

89. *What two numbers are those, whose difference, sum and product are to each other as are the numbers two, three and five respectively; that is, whose difference is to their sum as two to three, and whose sum is to their product as three to five?*

SOLUTION.

Put x for the greater number, and y for the less; then will their difference be $x - y$, their sum $x + y$, and their product yx ; and we shall have these two proportions productive of two equations, 1st, $x - y$ is to $x + y$ as 2 to 3, whence $3x - 3y = 2x + 2y$; 2d, $x + y$ is to yx as 3 to 5, whence $3yx = 5x + 5y$: the resolution follows;

Equ. 1st, $x - 5y = 0$.

Equ. 2d, $3yx - 5x - 5y = 0$.

Multiply the first equation by $3y - 5$, the coefficient of x in the second, and the product will be $3yx - 5x - 15yy + 25y = 0$; subtract this from the second equation, and you will have

Equ. 3d, $15yy - 30y = 0$; whence

Equ. 4th, $y = 2$, and

Equ. 5th, $x = 10$.

And the numbers 10 and 2 will answer the conditions of the problem.

Otherwise thus: by the first equation $x = 5y$; substitute therefore x instead of $5y$ in the second, and you will have $3yx - 5x - x = 0$; divide by x and you will have $3y - 5 - 1 = 0$, and $y = 2$, as before.

PROBLEM 63.

90. *It is required to find two numbers such, that if their difference be multiplied into their sum, the product will be five; but if the difference of their squares be multiplied into the sum of their squares, the product will be sixty five.*

SOLUTION.

Put x for the greater number, and y for the less; then will their difference be $x - y$, their sum $x + y$, and the product of their sum and difference multiplied together will be $x^2 - y^2$, by art. 11; then will $x^2 - y^2 = 5$ by the supposition, and $x^2 = 5 + yy$; square both sides, and you will have $x^4 = 25 + 10y^2 + y^4$: again, the difference of the squares of the two numbers sought is $x^2 - y^2$, and the sum of their squares $x^2 + y^2$, and the product of these two $x^4 - y^4$; therefore $x^4 - y^4 = 65$ by the supposition, and $x^4 = 65 + y^4$; but x^4 was before found equal to

$25 + 10y^2 + y^4$; therefore $25 + 10y^2 + y^4 = 65 + y^4$; whence $y^2 = 4$, and $y = 2$; substitute now 4 for y^2 in the first fundamental equation, which was $x^2 - y^2 = 5$, and you will have $x^2 - 4 = 5$, and $x = 3$; therefore the numbers sought are 3 and 2, which will answer the conditions.

PROBLEM 64.

91. *There is a certain number consisting of three places, whose digits are in arithmetical progression; if this number be divided by the sum of it's digits, the quotient will be forty eight; and lastly, if from the number be subtracted a hundred ninety eight, the digits will be inverted: I demand the number.*

N. B. Numbers are said to be in arithmetical progression, when they increase or decrease with equal differences: thus the numbers 5, 7 and 9 are said to be in arithmetical progression; as are also 9, 7 and 5. For the better understanding of the following solution of this problem, see problem 59.

SOLUTION.

Here I might put x , y and z for the three digits sought; but as these digits are said to be in arithmetical progression, and because it is esteemed more elegant to work with fewer unknown quantities than with more when it can conveniently be done, I shall put x and $x+y$ and $x+2y$ for the three digits sought; and since x and $x+y$ and $x+2y$ when added together make $3x+3y$, it is plain that $3x+3y$ will represent the sum of the digits: again, x in the hundred's place stands for $100x$, and $x+y$ in the ten's place stands for $10x+10y$, and the quantity $x+2y$ in the unit's place stands for itself; and these added together make $111x+12y$; therefore $111x+12y$ will represent the number sought: lastly, when the digits are inverted, $x+2y$ in the hundred's place will stand for $100x+200y$, and $x+y$ in the ten's place will stand for $10x+10y$, and x in the place of units will stand for itself; and these added together will make $111x+210y$; therefore $111x+210y$ will represent the number sought with it's digits inverted: these things being premised, the problem furnishes the two following equations, to wit,

$$\frac{111x+12y}{3x+3y} = 48, \text{ and}$$

$$111x+12y-198 = 111x+210y.$$

From the first of these equations, to wit, $\frac{111x+12y}{3x+3y} = 48$, we have $144x+144y=111x+12y$; therefore $33x+144y=12y$; therefore $33x+132y=0$; therefore (dividing by 33) $x+4y=0$, and $x=-4y$;

Art. 91, 92. PRODUCING SIMPLE EQUATIONS. 155

$-4y$; which is the result of the first equation: from the other equation, to wit, $111x + 12y - 198 = 111x + 210y$, we have $210y - 12y = -198$, whence $198y = -198$, and $y = -1$; but by the result of the former equation, $x = -4y$, that is, $-4 \times y$ or $-4 \times -1 = +4$; therefore x (the first digit toward the left hand) $= 4$; and since y , that is, $+y = -1$, the next digit $x + y$, will be $4 - 1 = 3$; and the last digit $x + 2y$ will be $4 - 2 = 2$; therefore the number sought is 432; the sum of whose digits is 9: for first, the digits 4, 3 and 2 are in arithmetical progression; 2dly, $\frac{432}{9} = 48$; and 3dly, $432 - 198 = 234$.

Though it might have been discovered at first sight from the very nature of this last problem, that the digits would not increase, but decrease from left to right, yet I supposed them to increase, only to shew, that though in all such cases we should make wrong suppositions, yet Algebra would always set us right: thus if the gain arising from any particular bargain is to be computed, we may put x for it, and so proceed as the conditions of the problem require; yet it is not impossible but that at the end of the operation x may come out negative; which would shew that what we supposed to be gain, was actually loss.

Some reflections concerning the conditions of problems.

92. In art. 70 it was said, that the equations by means whereof the values of unknown quantities are discovered, ought to be independent one of another; the reason of which assertion was, that if these equations were not independent, they must either be consequent one of another, or contradictory one to another; and all conclusions drawn from them would either be trifling, or absurd: for if the equations be consequent one of another, all you arrive to at last will be, that something is equal to itself; and if the equations be inconsistent one with another, you will find at last that some greater quantity is equal to a less: as for example, let the equations be $2x = 3y$, and $4x = 6y$; then it is plain, that this last equation is a consequence of the former, as being the double of the former; and if you should offer to resolve these two equations by any of the foregoing rules, you would come to no more at last than this, that $0 = 0$: again, let the equations be $2x = 3y$, and $4x = 6y + 7$; here it is plain, that this latter equation contradicts the former; for if $2x$ be equal to $3y$, then $4x$ ought to be equal to $6y$, and not to $6y + 7$; and if these two equations be resolved, or be attempted to be resolved by any of the foregoing rules, you will come at last to this absurdity, that $7 = 0$; and thus it very often happens that at the latter end of an operation,

equations are discovered to be consequential of, or inconsistent with one another, which at the beginning could not be so easily perceived.

I proceed now to give an example or two of problems wherein more than two unknown quantities are concerned.

PROBLEM 65.

93. *Three persons, A, B and C were talking of their money; says A to B and C, give me half of your money, and I shall have d; says B to A and C, give me a third part of your money, and I shall have d; says C to A and B, give me a fourth part of your money, and I shall have d: How much money had each?*

N. B. The letter *d* is here supposed to supply the place of some known quantity, which is left undetermined till the calculation is over.

SOLUTION.

Let *a*, *b* and *c* represent the money of *A*, *B* and *C* respectively, and we shall have these three fundamental equations;

$$a + \frac{b+c}{2} = d;$$

$$b + \frac{a+c}{3} = d; \text{ and}$$

$$c + \frac{a+b}{4} = d.$$

These equations, after due preparations according to art. 70, will stand thus;

$$\text{Equ. 1st, } 2a + b + c = 2d.$$

$$\text{Equ. 2d, } a + 3b + c = 3d.$$

$$\text{Equ. 3d, } a + b + 4c = 4d.$$

Subtract the first equation from twice the second, and you will have

$$\text{Equ. 4th, } * \quad 5b + c = 4d.$$

Subtract the third equation from the second, and you will have

$$\text{Equ. 5th, } * \quad 2b - 3c = -d.$$

Subtract five times the fifth equation from twice the fourth, and you will have

$$\text{Equ. 6th, } * \quad * \quad 17c = 13d.$$

$$\text{Equ. 7th, } * \quad * \quad c = \frac{13d}{17}.$$

Put this value for *c* in the fourth equation, and you will have $5b + c$, that is, $5b + \frac{13d}{17} = 4d$; therefore $85b + 13d = 68d$; therefore $85b = 55d$,

$= 55d$, and $b = \frac{55d}{85} = \frac{11d}{17}$; therefore

$$\text{Equ. 8th, } * b * = \frac{11d}{17}.$$

Put now the two values of b and c already found, instead of b and c in the first equation, and you will have $2a + b + c$, that is, $2a + \frac{11d + 13d}{17}$, or $2a + \frac{24d}{17} = 2d$; whence $34a + 24d = 34d$; and $34a = 10d$, and $a = \frac{10d}{34} = \frac{5d}{17}$; therefore

$$\text{Equ. 9th, } a * * = \frac{5d}{17}.$$

So that the numbers are at last found to be $a = \frac{5d}{17}$, $b = \frac{11d}{17}$, and $c = \frac{13d}{17}$; whence it follows, that if any number be put for d , that will admit of the number 17 for a divisor, the quantities a , b and c will come out in whole numbers: as if d be made equal to 17, the quantities a , b and c will be 5, 11 and 13 respectively; and the numbers will answer the conditions of the problem; for $5 + \frac{11 + 13}{2}$, or $5 + 12 = 17$; $11 + \frac{5 + 13}{3}$, or $11 + 6 = 17$; $13 + \frac{5 + 11}{4}$, or $13 + 4 = 17$.

Advertisment. I hope the reader does not need to be told, that the numbers a , b and c must always be understood to be of the same denomination with the number d ; as if the number d signifies so many guineas, the numbers a , b and c must also signify guineas; if shillings, shillings; if pence, pence; &c.

$$\text{Equ. 1st, } 2a + b + c = 2d.$$

$$2d, a + 3b + c = 3d.$$

$$3d, a + b + 4c = 4d.$$

$$4th, * 5b + c = 4d.$$

$$5th, * 2b - 3c = -d.$$

$$\text{Equ. 6th, } * * 17c = 13d.$$

$$7th, * * c = \frac{13d}{17}.$$

$$8th, * b * = \frac{11d}{17}.$$

$$9th, a * * = \frac{5d}{17}.$$

A SCHOLIUM.

94. Of the foregoing equations, the first, second and third, wherein the quantity a is concerned, may be called equations of the first rank; the fourth and fifth, wherein the quantity b is concerned, and out of which the quantity a is excluded, may be called equations of the second rank; the sixth, wherein c is concerned, and out of which both a and b are excluded, may be called an equation of the third rank; and so on, were there ever so many unknown quantities.

Whenever the equations of any particular rank are given or found, in order to derive from thence equations of an inferior rank, the Analyst is at liberty to combine these first equations by pairs as he pleases, provided he does but observe these two things; first, that every equation of the given rank be some time or other coupled with some other equation of the same set, so as that no equation be left out of the account; secondly, that in every particular combination, one of the equations be such as was never made use of in any combination before, and the other such as hath been concerned in some combination before, excepting the first pair. It is not to be denied but that the artist may, if he pleases, vary sometimes from this last precept; but if he always observes it, it will be altogether as well.

PROBLEM 66.

95. *Three persons, A, B and C were talking of their money; says A to B and C, give me e out of your stock, and I shall have twice as much as you will have left; says B to A and C, give me e out of your stock, and I shall have three times as much as you will have left; says C to A and B, give me e out of your stock, and I shall have four times as much as you will have left: How much money had each?*

SOLUTION.

Put a , b and c for the money of A , B and C respectively, and you will have these three fundamental equations;

$$a + e = 2b + 2c - 2e.$$

$$b + e = 3a + 3c - 3e.$$

$$c + e = 4a + 4b - 4e.$$

Which being prepared, stand thus;

$$\text{Equ. 1st, } a - 2b - 2c = -3e.$$

$$\text{Equ. 2d, } 3a - b + 3c = 4e.$$

$$\text{Equ. 3d, } 4a + 4b - c = 5e.$$

Subtract

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Subtract three times the first equation from the second, and you will have

$$\text{Equ. 4th, } * \quad 5b + 9c = 13e.$$

Subtract four times the first equation from the third, and you will have

$$\text{Equ. 5th, } * \quad 12b + 7c = 17e.$$

Subtract five times the fifth equation from twelve times the fourth, and you will have

$$\text{Equ. 6th, } * \quad * \quad 73c = 71e; \text{ whence}$$

$$\text{Equ. 7th, } * \quad * \quad c = \frac{71e}{73}.$$

Put this value instead of c in the fourth equation, and you will have

$$5b + 9c, \text{ that is, } 5b + \frac{639e}{73} = 13e; \text{ whence } 365b + 639e = 949e;$$

$$\text{therefore } 365b = 310e, \text{ and } b = \frac{310e}{365} = \frac{62e}{73}; \text{ therefore}$$

$$\text{Equ. 8th, } * \quad b \quad * = \frac{62e}{73}.$$

$$\text{Therefore } b + c = \frac{133e}{73}, \text{ and } 2b + 2c = \frac{266e}{73}, \text{ and } -2b - 2c = -$$

$$\frac{266e}{73}; \text{ substitute therefore } -\frac{266e}{73} \text{ instead of } -2b - 2c \text{ in the first equa-}$$

$$\text{tion, and you will have } a - \frac{266e}{73} = -3e; \text{ therefore } 73a - 266e =$$

$$-219e; \text{ therefore}$$

$$\text{Equ. 9th, } a \quad * \quad * = \frac{47e}{73}.$$

$$\text{So that at last, } A's \text{ money is discovered to be } \frac{47e}{73}, B's \frac{62e}{73}, \text{ and } C's$$

$$\frac{71e}{73}; \text{ make } e = 73, \text{ and then } A's \text{ money will be } 47, B's 62, \text{ and } C's$$

71, as will easily appear thus; A 's stock is 47, B 's and C 's together make 133; add 73 to 47, and take it from 133, and A will have 120, and B and C will have 60; and $60 \times 2 = 120$: again, B 's money is 62, and A 's and C 's together make 118; subtract 73 from 118, and add it to 62, and B will have 135, and A and C will have 45; and $45 \times 3 = 135$: lastly, C 's stock is 71, and B 's and A 's make 109; subtract 73 from 109, and add it to 71, and C will have 144, and A and B 36; and $36 \times 4 = 144$.

Equ,

$$\text{Equ. 1st, } a - 2b - 2c = -3e.$$

$$2d, 3a - b + 3c = 4e.$$

$$3d, 4a + 4b - e = 5e.$$

$$4\text{th, } * 5b + 9c = 13e.$$

$$5\text{th, } * 12b + 7c = 17e.$$

$$6\text{th, } * * 73c = 71e.$$

$$\text{Equ. 7th, } * * c = \frac{71e}{73}.$$

$$8\text{th, } * b * = \frac{62e}{73}.$$

$$9\text{th, } a * * = \frac{47e}{73}.$$

PROBLEM 67.

96. It is required to find four numbers, a, b, c and d so related to each other, that a with half the sum of all the rest may make f ; that b with a third part of all the rest may make f ; that c with a fourth part of all the rest may make f ; and that d with a fifth part of all the rest may make f .

SOLUTION.

The fundamental equations of this problem are

$$a + \frac{b+c+d}{2} = f.$$

$$c + \frac{a+b+d}{4} = f.$$

$$b + \frac{a+c+d}{3} = f.$$

$$d + \frac{a+b+c}{5} = f.$$

From which equations duly prepared, are derived the following;

$$\text{Equ. 1, } 2a + b + c + d = 2f.$$

$$2, a + 3b + c + d = 3f.$$

$$3, a + b + 4c + d = 4f.$$

$$4, a + b + c + 5d = 5f.$$

$$5, * 5b + c + d = 4f.$$

$$6, * 2b - 3c * = -f.$$

$$7, * * 3c - 4d = -f.$$

$$8, * * 17c + 2d = 13f.$$

$$9, * * * 74d = 56f.$$

$$10, * * * d = \frac{28f}{37}.$$

$$11, * * c * = \frac{25f}{37}.$$

$$12, * b * * = \frac{19f}{37}.$$

$$13, a * * * = \frac{1f}{37}.$$

Twice the 2d — the first gives

The 2d — the 3d gives

The 3d — the 4th gives

Twice the 5th — 5 times the 6th gives

Thrice the 8th — 17 times the 7th gives

This last gives

The 7th and 10th give

The 6th and 11th give

The 1st, 12th, 11th and 10th give

Make

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Make $f=37$, and the numbers a, b, c, d , will come out 1, 19, 25, 28 respectively.

PROBLEM 68.

97. Four gamesters, A, B, C and D, each with a different stock of money about him, but as yet unknown, sit down to play; during which engagement, A wins half of B's first stock, B wins a third part of C's, C a fourth part of D's, and D a fifth part of A's; after this they all rise with the same sum of money about them, to wit, g ; which g , though a known quantity, is not supposed to be determined till the operation is over: It is required to determine the several stocks of money with which the gamesters A, B, C, D, began to play; I mean with respect to the known quantity g .

Various solutions may be given of this problem; but that which seems most proper to be inserted here, is the following

SOLUTION.

Let a, b, c, d , represent the respective unknown sums with which A, B, C and D began to play; then if A had only lost a fifth part of his first stock to D, and had won nothing of B, his last stock would have been $\frac{a}{1} - \frac{a}{5}$, or $\frac{4a}{5}$; but according to the problem, A did not only lose a fifth part of his stock to D, but also won half of B's first stock; therefore A's last sum when the play was ended was $\frac{4a}{5} + \frac{b}{2}$; but the problem informs us, that when the play was ended, A's last sum was g ; whence we have this equation, $\frac{4a}{5} + \frac{b}{2} = g$; therefore $4a + \frac{5b}{2} = 5g$; therefore $8a + 5b = 10g$: in like manner the first and second conditions of the problem furnish this equation, $\frac{b}{1} - \frac{b}{2} + \frac{c}{3} = g$, or $\frac{b}{2} + \frac{c}{3} = g$; which freed from fractions like the former, becomes $3b + 2c = 6g$: the second and third conditions of the problem furnish this equation, $\frac{c}{1} - \frac{c}{4} + \frac{d}{3} = g$, or $\frac{2c}{3} + \frac{d}{4} = g$; which freed from fractions like the rest, becomes $8c + 3d = 12g$: lastly, the last and first conditions of the problem also considered together, give this equation, $\frac{3d}{4} + \frac{a}{5} = g$; which in integral terms, stands thus, $15d + 4a = 20g$: so that

X

the

the fundamental equations of this problem when freed from fractions will stand thus;

$$8a + 5b = 10g.$$

$$3b + 2c = 6g.$$

$$8c + 3d = 12g.$$

$$4a + 15d = 20g.$$

Amongst these equations it is plain there are but two of the first rank, to wit, the first, $8a + 5b = 10g$, and the last, $4a + 15d = 20g$; the second fundamental equation in order, to wit, $3b + 2c = 6g$ is an equation of the second rank; and the third in order, $8c + 3d = 12g$ is an equation of the third rank; therefore in the resolution of this problem, these fundamental equations must not be written all together one under another, as in the former examples, but must be reserved for their proper ranks, thus;

$$\text{Equ. 1st, } 8a + 5b \quad * \quad * = 10g.$$

$$2d, \quad 4a \quad * \quad * + 15d = 20g.$$

Now as there are no other equations but these two of the first rank, proceed on to equations of the second rank thus; according to art. 70, subtract twice the second equation from the first, and there will remain $5b - 30d = -30g$; whence dividing by 5, you will have $b - 6d = -6g$ for a third equation, which ought to be placed regularly under the second, thus;

$$\text{Equ. 3d, } \quad * \quad b \quad * - 6d = -6g.$$

But as this last equation is an equation of the second rank, it will now be proper to draw out the equation of the second rank hitherto kept in reserve in the problem, to wit, $3b + 2c = 6g$, and to place it under the rest for a fourth equation, thus;

$$\text{Equ. 4th, } \quad * \quad 3b + 2c \quad * = 6g.$$

Subtract now three times the third equation from the fourth, that is, subtract $* 3b * - 18d = -18g$ from $* 3b + 2c * = 6g$, and there remains $2c + 18d = 24g$; whence dividing by 2, you will have

$$\text{Equ. 5th, } \quad * \quad * \quad c + 9d = 12g.$$

This is an equation of the third rank, and therefore it will now be proper to draw out the last equation of the fundamental ones given in the problem, which is of the same rank with this, and to place it under this for a sixth equation, thus;

$$\text{Equ. 6th, } \quad * \quad * \quad 8c + 3d = 12g.$$

Subtract this sixth equation from eight times the fifth, that is, $* * 8c + 3d = 12g$ from $* * 8c + 72d = 96g$, and there remains $69d = 84g$; whence dividing by 3, we have

$$\text{Equ. 7th, } \quad * \quad * \quad * \quad 23d = 28g; \text{ and}$$

$$\text{Equ. 8th, } \quad * \quad * \quad * \quad d = \frac{28g}{23}.$$

Sub-

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Substitute this value for d in the fifth equation, and instead of $c + 9d = 12g$, you will now have $c + \frac{252g}{23} = 12g$; whence $23c + 252g = 276g$; and $23c = 24g$; and

$$\text{Equ. 9th, } * * c * = \frac{24g}{23}.$$

Substitute the value of d found in the 8th equation instead of d in the 3d, where we had $b - 6d = -6g$, and you will have $b - \frac{168g}{23} = -6g$; whence $23b - 168g = -138g$; and $23b = 30g$; and

$$\text{Equ. 10th, } * b * * = \frac{30g}{23}.$$

Substitute this value for b in the first equation, $8a + 5b = 10g$, and you will have $8a + \frac{150g}{23} = 10g$; whence $184a + 150g = 230g$; therefore $184a = 80g$, and $a = \frac{80g}{184} = \frac{10g}{23}$; whence we have

$$\text{Equ. 11th, } a * * * = \frac{10g}{23}.$$

So that at last all the quantities a, b, c, d , come to be known with respect to the quantity g ; for $a = \frac{10g}{23}$, $b = \frac{30g}{23}$, $c = \frac{24g}{23}$, and $d = \frac{28g}{23}$: make $g = 23$, and the quantities will all come out in whole numbers thus; $a = 10$, $b = 30$, $c = 24$, and $d = 28$; and they will answer the conditions of the question; for at this rate

A 's last sum will be $10 - 2 + 15 = 23$;

B 's last sum will be $30 - 15 + 8 = 23$;

C 's last sum will be $24 - 8 + 7 = 23$;

and D 's last sum will be $28 - 7 + 2 = 23$.

Equ. 1st, $8a + 5b$	$*$	$*$	$= 10g.$	Equ. 8th, $***$	$d = \frac{28g}{23}.$
2d, $4a$	$*$	$*$	$+ 15d = 20g.$	9th, $***$	$c = \frac{24g}{23}.$
3d, $*$	b	$*$	$- 6d = -6g.$	10th, $* b *$	$= \frac{30g}{23}.$
4th, $*$	$3b + 2c$	$*$	$= 6g.$	11th, $a ***$	$= \frac{10g}{23}.$
5th, $*$	$*$	$c + 9d = 12g.$			
6th, $*$	$*$	$8c + 3d = 12g.$			
7th, $*$	$*$	$23d = 28g.$			

The solution here given is designed to shew the learner how to proceed in cases where all his fundamental equations are not of the same rank; for otherwise, this problem is capable of a much more elegant solution by the help of one letter only, as we shall find, if to avoid fractions, we put x for the money D won of A , thus;

D won of A ,	x ;
A 's first stock,	$5x$;
Left A after his loss to D ,	$4x$;
A won of B ,	$g - 4x$;
B 's first stock,	$2g - 8x$;
Left B after his loss to A ,	$g - 4x$;
B won of C ,	$4x$;
C 's first stock,	$12x$;
Left C after his loss to B ,	$8x$;
C won of D ,	$g - 8x$;
D 's first stock,	$4g - 32x$;
Left D after his loss to C ,	$3g - 24x$.

D won of A , $24x - 2g$; for the remainder of D 's first stock after his loss to C was $3g - 24x$, and therefore if D had won nothing of A , this quantity $3g - 24x$ would have been D 's last sum; but by the problem it appears that D 's last sum was g ; therefore the money D won of A must be the excess of g above $3g - 24x$; subtract therefore $3g - 24x$ from g , and the remainder $24x - 2g$ will be the money D won of A ; and after the same manner were all the other winnings determined: here then we have two expressions for the money D won of A , to wit, x according to the supposition, and $24x - 2g$ from the nature of the question; therefore $24x - 2g = x$; and $x = \frac{2g}{23}$: this being determined, the first stocks of all the gamesters will be determined by the positions, thus:

A 's first stock, $5x = \frac{10g}{23}$.	C 's $12x = \frac{24g}{23}$.
B 's $2g - 8x = \frac{30g}{23}$.	D 's $4g - 32x = \frac{28g}{23}$.

THE

THE
ELEMENTS of ALGEBRA

BOOK III.

*Some observations tending to the investigation of the
rule for extracting the square root.*

98. **H**AVING promised somewhere that I would take the first opportunity to account for the common method of extracting the square root, I cannot think of any place more proper than this for the performance of my promise, not only as the practice of this rule grows upon our hands in the resolution of quadratic equations, and therefore ought not, if possible, to be left any longer undemonstrated, but more especially considering that by this time our young Analyst may be reasonably supposed to be pretty well acquainted with some of the first rudiments of his art, which may enable him to follow me with more ease to himself, and perhaps to me too, than could possibly have been done at our first setting out. But here I must not omit to advertise my reader, that if a whole number, or a mixt number consisting of a whole number and decimal parts, be not a square, its square root will consist of an integral part, and of an infinite series of decimal fractions: but if a rule can be found out for determining the integral part, it will be sufficient, because all the parts of the square root of any number may be considered as integral till the operation is over, since it is not till then, that the distinction needs to be made between integral and decimal parts; see introduct. art. 24.

For the better effecting what I here propose, I shall lay down the following observations, which the learner must attend to, if he expects to go through the following demonstration; and if there be still any difficulties he may meet with or thinks he meets with in the application of these observations, the best advice I can give him here, as well as in many other parts of this book, is to read the demonstration over and over again, by which means all the steps will become more familiar to him, and he will be the better able to put them together, in order to digest the whole.

O B S E R -

OBSERVATION 1.

If any number, as $a + x$, consisting of two parts a and x , be squared, the product will be $aa + 2ax + xx = aa + 2a + x \times x$.

OBSERVATION 2.

$1 \times 1 = 1$; therefore $10 \times 10 = 100$, and $100 \times 100 = 10000$, and $1000 \times 1000 = 1000000$, &c: whence I infer, that if any number consists of one or two places, it's square root, or at least the integral part of it, will consist but of one place; if a number consists of three or four places, the integral part of it's square root will consist of two places; if a number consists of five or six places, the integral part of it's square root will consist of three places, &c: thus the integral part of the square root of this number of five places, to wit, 56644 will consist of three places; for the number 56644, lies between 10000, whose square root is 100, and 1000000, whose square root is 1000; therefore the integral part of the square root of the number 56644 must lie between 100 and 1000, and consequently must consist of three places.

OBSERVATION 3.

From the last observation it follows, that if of any number proposed, a point be put over the place of units, and another over the next place but one to the left hand, I mean that of hundreds, and another again over the next place but one to that, to wit, of ten thousands, and so on alternately; the number of points will discover the number of places of which the integral part of the square root of the proposed number consists: thus if the number 56644 be so pointed, it will stand thus, $\dot{5}6\dot{6}4\dot{4}$; and the three points shew, that the square root of this number, or the integral part of it at least, consists of three places.

OBSERVATION 4.

Supposing all things as in the last observation, if any number be advanced one point, that is, two places higher to the left hand, by adding cyphers, or by taking in decimals from the right hand, the figures expressing it's square root will still be the same, but will be advanced one place higher; if the number be advanced two points, the square root will be advanced two places; if three points, three places, &c: thus if the square root of the number 576 be 24, that of the number 57600 will be 240, that of 5760000 will be 2400, &c: thus again, if the square

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square root of the number 5.6644 be 2.38 , the square root of the number 566.44 will be 23.8 , and that of the number 56644 will be 238 , &c: whoever would be thoroughly convinced of this, as well as of the reason of the thing, may easily satisfy himself by squaring the numbers 2.38 , 23.8 , 238 .

Note, that by squaring a number is here as well as every where else meant, multiplying it into itself, and not extracting it's square root, as young beginners through inadvertency are very often apt to mistake it.

OBSERVATION 5.

From the last observation it follows, that if the square root of any number, suppose of 5.6644 lies between 2 and 3, the square root of 566.44 will lie between 20 and 30; and that of 56644 , between 200 and 300, &c: thus again, if the square root of 566.44 lies between 23 and 24, the square root of 56644 will lie between 230 and 240: this is plain; for suppose the square root of 566.44 to be 23.8 , then by the last observation, the square root of 56644 will be 238 ; and as 23.8 lies between 23 and 24, so 238 lies between 230 and 240.

The investigation of the rule for extracting the square root.

99. These things being observed, let now some number be proposed, such as is the number 56644 ; and let us see whether we cannot investigate the square root of this number by pure dint of reason, without any regard to, or helps from the common method. This number then being pointed according to the third observation, I shall first begin with the number 5, belonging to the first point to the left hand, setting aside all the other figures, thus: the number 5 is itself no square number, therefore I subtract the number 4 which is the nearest square number less than 5, from 5, and there remains 1; and as 2 is the square root of 4, I conclude, that of all the infinite series representing the square root of the number 5, the first term, or the integral part will be 2: again, as the square of 2 is the nearest square number less than 5, which is no square, it follows, that the square of 3 will be the nearest square number greater than 5, and consequently that the square of 3, whatever it is, cannot be less than 6; whence I conclude in the next place, that the number 2 will be the first term of the square root, not only of the number 5, but also of any other number between 5 and 6 at least; therefore 2 will be the first figure expressing the square root of the number 5.66 , as also of the number 5.6644 ; therefore by the fourth observation, 2 will be the first

figure

figure of the square root of the number 56644; which is one considerable point gained: now to discover the next figure in the root, I consider the number 566 belonging to the two first dots to the left hand, setting aside all the other figures, thus; it has already been shewn that the first figure expressing the square root of the number 5.66 is 2; therefore the square root of the number 5.66 lies between 2 and 3; therefore by the fifth observation, the square root of the number 566 lies between 20 and 30; let then $20+x$ represent the integral part of the square root of the number 566, the letter x standing for some whole number in the place of units; then it is plain, that the square of $20+x$ must either be precisely equal to 566 if 566 be an exact square, or else it must be the nearest square number less than 566; but the square of $20+x$ by the first observation is $20 \times 20 + 40 + x \times x$; therefore $20 \times 20 + 40 + x \times x$ is either equal to, or less than 566; subtract the square of 20 from both sides, by observing, that as before, 2×2 subtracted from 5 left 1, so now 20×20 subtracted from 500 will leave 100, and the same subtracted from 566, will leave 166; therefore $40 + x \times x$ must either be equal to, or less than 166, according as the number 566 is, or is not a perfect square; and therefore if $40 + x \times x$ be less than 166, the difference will be the excess of the number 566 above the nearest number less than it that is an exact square: here we see, by the bye, why the number 166 is commonly called the resolvend, to wit, because it must be resolved into two factors, x and $40+x$, the product of whose multiplication must either be equal to 166, or else it must be the nearest product of the kind less: but to proceed; as it has been shewn already that the product $40+x \times x$ is not to exceed 166, it follows, that the product $40x$ must be less than 166, and consequently that x must be less than the quotient of 166 divided by 40, or of 16.6 divided by 4; but the quotient of 16.6 divided by 4 lies between the two whole numbers 4 and 5; therefore x is less than 5, and 4 is the greatest whole number that can be supposed equal to x ; let us then suppose x equal to 4, and let us see what will be the consequence; now if $x=4$, we shall have $40+x=44$, and $40+x \times x=44 \times 4=176$; therefore the supposition of $x=4$ was wrong, because $40+x \times x$ ought not to exceed 166; let us then suppose in the next place that $x=3$; then we shall have $40+x=43$, and $40+x \times x=43 \times 3=129$, which is less than 166, and therefore not inconsistent with the foregoing conditions; and since x must be taken equal to the greatest whole number the foregoing conditions will admit of, it follows, that x must be equal to 3, and consequently that $20+x$,

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 or the integral part of the square root of the number 566 is 23; and
 since $40 + x \times x$ or 129, subtracted from the resolvend 166 leaves 37,
 it follows, that 23×23 subtracted from the number 566 will also leave
 37: again, as the square of 23 is the nearest square number less than
 566 which is no square, the square of 24 must be greater than 566, and
 consequently cannot be less than 567; therefore the square root of the
 number 566.44 must necessarily lie between 23 and 24; therefore by
 the fifth observation, the square root of the number 56644 must neces-
 sarily lie between 230 and 240; which is another step made in the ap-
 proximation: let us now suppose $230 + x$ to be the square root, or the
 integral part of the square root of the number 56644; then by the first
 observation we shall have $230 \times 230 + 460 + x \times x$ either equal to, or
 less than 56644: subtract 230×230 from both sides, by observing, that
 as before 23×23 subtracted from 566 left 37, so now 230×230 sub-
 tracted from 56600 will leave 3700; and the same subtracted from
 56644 will leave 3744; therefore $460 + x \times x$ must not exceed 3744;
 therefore $460x$ must be less than 3744; therefore x must be less than
 the quotient of 3744 divided by 460; but the quotient of 3744 divided
 by 460, or (which is pretty much the same thing, especially with re-
 spect to the integral part) the quotient of 374 divided by 46 lies between
 the two whole numbers 8 and 9; therefore 8 is the greatest whole num-
 ber that can be supposed equal to x : let us then suppose $x = 8$, and we
 shall have $460 + x = 468$, and $460 + x \times x = 468 \times 8 = 3744$, which
 is just consistent with the abovementioned conditions, and the number
 238 is the exact square root of the number proposed 56644. Q. E. I.

Let now any one extract the square root of the same number 56644
 according to the common practice, and he will easily see, that the rule he
 there uses is nothing else but a general observation drawn from this or
 the like investigation.

*The foundation of the common rule for extracting the
 cube root.*

100. The cube of $a + x$ is $a^3 + 3a^2x + 3ax^2 + x^3$, where $3a^2x +$
 $3ax^2 + x^3$ is the resolvend; make $3a + x = b$, and you will have $3ax +$
 $xx = bx$, and $3a^2 + 3ax + xx = 3a^2 + bx$; and $3a^2x + 3ax^2 + x^3$,
 or the resolvend $= 3a^2 + bx \times x$; therefore the cube of $a + x$ is $a^3 +$
 $3a^2 + bx \times x$: this answers to the first observation in the last article but
 one, and is the foundation of the rule for extracting the cube root;
 and whosoever applies the reasoning of the two last articles to the present
 case,

case, he will either be able of himself to form a rule for extracting the cube root, or at least he will be the better able to understand and remember the common rule as it is delivered in books of Arithmetic.

N. B. All the difficulty in extracting either the square or cube root, is in obtaining the two first figures of the root, wherein tryals are sometimes required; the rest of the process is more certain.

Of the composition and resolution of a square raised from a binomial root.

101. Hitherto we have been chiefly concerned in simple equations: it is now high time to apply ourselves to the resolution of quadratics; in order to which, something must be said concerning the nature of a binomial, upon which that resolution entirely depends.

Now a binomial (at least as it is here used) is a quantity consisting of two parts or members, connected together by the sign $+$ or $-$, as $x+a$, $x-a$, $x+\frac{b}{2}$, $x-\frac{b}{2}$; and a square raised from a binomial root is no-

thing else but the square of such a quantity: thus the square of $x+\frac{b}{2}$ is $xx+\frac{bx}{2}+\frac{bb}{4}$, and that of $x-\frac{b}{2}$ is $xx-\frac{bx}{2}+\frac{bb}{4}$.

$$\begin{array}{r} x+\frac{b}{2} \\ x+\frac{b}{2} \\ \hline \frac{bx}{2} \quad \frac{bb}{4} \\ x^2+\frac{bx}{2}+\frac{bb}{4} \\ \hline \frac{bx}{2} \\ +\frac{b}{2} \\ \hline \end{array}$$

$$\begin{array}{r} x-\frac{b}{2} \\ x-\frac{b}{2} \\ \hline \frac{bx}{2} \quad \frac{bb}{4} \\ x^2-\frac{bx}{2}+\frac{bb}{4} \\ \hline \frac{bx}{2} \\ -\frac{b}{2} \\ \hline \end{array}$$

$$x^2+\frac{2bx}{2}+\frac{bb}{4}; \text{ that is, } x^2+bx+\frac{bb}{4}. \quad x^2-\frac{2bx}{2}+\frac{bb}{4}; \text{ that is, } x^2-bx+\frac{bb}{4}.$$

The difference betwixt these two squares arises from the different sign of b ; and that only affects the second member; for the third member $\frac{bb}{4}$ will be the same, whether the quantity b be affirmative or negative; therefore if those cases be thrown into one, it will stand thus: *The square of*

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of $x \pm \frac{b}{2}$ is $xx \pm bx + \frac{bb}{4}$; to wit, $+bx$ when the root is $x + \frac{b}{2}$, and $-bx$ when the root is $x - \frac{b}{2}$. Now of the three members that compose this square, the first xx is the square of x , the second $\pm bx$ is the root of that square multiplied into the coefficient $\pm b$; for the root of xx is x , and $xx \pm b = \pm bx$; the third and last member $\frac{bb}{4}$, is the square of $\pm \frac{b}{2}$, that is, the square of half the coefficient of the second member; whence may be deduced the two following observations.

OBSERVATION I.

Whenever we meet with a quantity consisting of two members, as $xx \pm bx$, whereof one, as xx , is a square, and the other $\pm bx$ is the root of that square multiplied into some given coefficient $\pm b$; whenever I say we meet with such a quantity, it may be considered as an imperfect square raised from a binomial root, and may easily be compleated by adding $\frac{bb}{4}$, that is, by adding the square of half the coefficient of x in the second term: thus $xx + 6x$ when compleated, becomes $xx + 6x + 9$; $xx - 8x$ when compleated becomes $xx - 8x + 16$; $xx + 3x$ when compleated becomes $xx + 3x + \frac{9}{4}$; for here the coefficient being 3, its half will be $\frac{3}{2}$, and the square of this will be $\frac{9}{4}$; again, $xx + \frac{2x}{3}$ when compleated becomes $xx + \frac{2x}{3} + \frac{1}{9}$; for here the second term is $\frac{2x}{3}$, and therefore the coefficient of x is $\frac{2}{3}$ by art. 70; but the half of $\frac{2}{3}$ is $\frac{1}{3}$, and the square of this is $\frac{1}{9}$; again, $xx - \frac{5x}{6}$ when compleated becomes $xx - \frac{5x}{6} + \frac{25}{144}$; for here the coefficient is $-\frac{5}{6}$, whose half is $-\frac{5}{12}$, and the square of this is $+\frac{25}{144}$; lastly, $xx - \frac{bx}{a}$ when compleated becomes $xx - \frac{bx}{a} + \frac{bb}{4aa}$; for here the coefficient is $-\frac{b}{a}$, its half $-\frac{b}{2a}$, and the square of this is $\frac{bb}{4aa}$.

OBSERVATION 2.

In the second place it may be observed, that the root of such a square when compleated, that is, the root of $xx \pm bx + \frac{bb}{4}$ will always be $x \pm \frac{b}{2}$, that

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is,

is, it will always be the square root of the first member, together with half the coefficient of the second: thus the square root of $xx+6x+9$ will be $x+3$; that of $xx-8x+16$ will be $x-4$; that of $xx+3x+\frac{9}{4}$ will be $x+\frac{3}{2}$; that of $xx+\frac{2x}{3}+\frac{1}{9}$ will be $x+\frac{1}{3}$; that of $xx-\frac{5x}{6}+\frac{25}{144}$ will be $x-\frac{5}{12}$; and lastly, that of $xx-\frac{bx}{a}+\frac{bb}{4aa}$ will be $x-\frac{b}{2a}$.

The common form to which all quadratic equations ought to be reduced in order to be resolved.

102. Since an affected quadratic equation, as we have elsewhere defined it (art. 23,) is an equation consisting of three different sorts of quantities; one sort wherein the square of the unknown quantity is concerned, another sort wherein the unknown quantity is simply concerned, and a third sort wherein it is not concerned at all; it follows, that all quadratic equations whatever may be reduced to this form, *viz.* $Axx=Bx+C$; wherein A , B and C denote known integral quantities whether affirmative or negative, and x the quantity unknown, the sign $+$ on the latter side of the equation $Bx+C$, signifying no more than that the two quantities Bx and C are to be added together according to the common rules of addition, whether they be both affirmative, or both negative, or one affirmative and the other negative: this will easily be allowed, if it be considered, that quadratic equations, like all others, may be freed from fractions after the same manner as simple equations; and when that is done, there needs no more at most, than a bare transposition of the terms to reduce them to the form above described: we shall however give some examples of the reduction of quadratic equations to this form amongst those that follow.

A general theorem for resolving all quadratic equations.

103. This preparation being made, let now some general quadratic equation be proposed to be resolved, with which all particular equations may afterwards be compared, and by means whereof those equations may be more readily resolved; as for example, let the general equation in the last article be proposed, to wit, $Axx=Bx+C$; and let it be proposed to find the value or values of x in this equation: here transposing Bx , I have $Axx-Bx=C$; and then dividing by A in order to
free

free xx the highest power of x from it's coefficient, I have $xx - \frac{Bx}{A} = \frac{C}{A}$; this done, I consider the first side $xx - \frac{Bx}{A}$ as an imperfect square raised from a binomial root; and accordingly I compleat that square by art. 101, to wit, by adding $\frac{BB}{4AA}$, that is, by adding the square of half the coefficient of the second term; but if $\frac{BB}{4AA}$ must be added to the first side of the equation to compleat the square, it must also be added to the other side to preserve the equality, otherwise by an unequal addition, the equation would be destroyed: this equal addition then being made, the equation will stand thus, $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB}{4AA} + \frac{C}{A}$; but the two fractions $\frac{BB}{4AA}$ and $\frac{C}{A}$ when thrown into one, give $\frac{ABB + 4AAC}{4AAA}$, which dividing by A , gives $\frac{BB + 4AC}{4AA}$; therefore $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB + 4AC}{4AA}$; therefore the square root of one side will be equal to the square root of the other; but the square root of the fraction $\frac{BB + 4AC}{4AA}$, at least as it here stands in letters, cannot be extracted, because, though the denominator $4AA$ be a square, yet there is no literal quantity whatever which being multiplied into itself will produce $BB + 4AC$; therefore to put this numerator into the form of a square, let us suppose $BB + 4AC = ss$; and then the equation will stand thus, $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{ss}{4AA}$; but the square root of $xx - \frac{Bx}{A} + \frac{BB}{4AA}$ is $x - \frac{B}{2A}$ by art. 101; and the square root of $\frac{ss}{4AA}$ is $\pm \frac{s}{2A}$ for a reason formerly given, to wit, because $\frac{-s}{2A}$ when multiplied into itself will produce $\frac{+ss}{4AA}$ as well as $\frac{+s}{2A}$ and therefore by the very definition of the square root, the former quantity has as good a right to be styled the square root of $\frac{ss}{4AA}$ as the latter; therefore this equation will

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now be reduced to a simple one, and will stand thus, $x - \frac{B}{2A} = \pm \frac{s}{2A}$;

therefore $x = \frac{B \pm s}{2A}$, that is, $x = \frac{B+s}{2A}$, and $x = \frac{B-s}{2A}$. Q. E. I.

Thus we see that every quadratic equation necessarily admits of two numbers or roots (as they are called) which will equally answer the condition of the equation, that is, either of which being put equal to x , will make the two sides of the equation equal one to the other; and these two roots in all arts and sciences where quadratic equations are concerned, are of equal estimation, whether they be affirmative or negative, or one be affirmative and the other negative: as for example, in Geometry, if a line drawn from any point towards the right hand be considered as affirmative, a line drawn from the same point to the left hand ought to be considered as negative; for let AB be any line drawn from the fixt point A to the point B on the right hand, and then imagine the point B to move towards A ; here then it is plain that the nearer B approaches towards A , the less will be the affirmative line AB ; when the point B coincides with A , the line AB must be looked upon as nothing, and therefore when the point B by a continuation of it's motion has passed through A , so as to lye on the left hand of A , the line AB ought now to be looked upon as negative, having passed from something through nothing into negation; and yet a line of this negative kind is as true a line as any of the affirmative kind; and therefore the negative roots of quadratic equations which exhibit negative lines, ought to be of equal estimation with the affirmative roots that exhibit affirmative lines; and the same will be the case (I say) of all other arts and sciences where quadratic equations are concerned: but in common life, where negative quantities have no place, the affirmative roots of quadratic equations are only allowed of in the resolution of problems, the negative ones being for the most part excluded.

N. B. 1st, The root of any quantity whether in numbers or letters, that cannot be expressed, is called a furd: thus $\sqrt{3}$ is a furd, and so also is $\sqrt{BB+4AC}$; and it was for this reason that I made $\sqrt{BB+4AC} = s$, or which is all one, $BB+4AC = ss$.

2^{dly}, The quantity C and consequently $4AC$ will sometimes be negative; in which case the quantity ss , or $BB+4AC$ must be looked upon as the sum of the affirmative quantity BB and the negative one $4AC$ when added together according to the common rules of addition.

3^{dly}, In many of the following examples, the learner must be very careful to form a right estimation of negative quantities: thus for instance, if x , that is, $+x = -3$, he must make $4x$, or $+4x = -12$; but

but he must make $-4x$, or $-4x-3=+12$; so likewise $-x$, or $-1x$, or $-1x-3$ must be made equal to $+3$, &c.

A synthetical demonstration of the foregoing theorem.

104. In the last article it was demonstrated analytically, that if Axx be equal to $Bx+C$, then x must necessarily be equal both to $\frac{B+s}{2A}$, and to $\frac{B-s}{2A}$, supposing ss to be equal to $BB+4AC$. Now it may not be improbable but that the learner, especially if he has any taste or genius, may have a curiosity to see the same demonstrated again synthetically, that is, to see it demonstrated, that if x be made equal to $\frac{B+s}{2A}$, or $\frac{B-s}{2A}$, then Axx must necessarily be equal to $Bx+C$: it is therefore to gratify the learner in this particular, that I have added the following demonstration.

CASE 1st.

Let $x = \frac{B+s}{2A}$; then you will have $xx = \frac{BB+2Bs+ss}{4AA}$; multiply both sides by A , and you will have Axx (or one side of the general equation) equal to $\frac{BB+2Bs+ss}{4A}$; for a fraction may be multiplied by dividing the denominator, as well as by multiplying the numerator: again, since $x = \frac{B+s}{2A}$, you will have $Bx = \frac{BB+Bs}{2A}$; double both the numerator and denominator of this last fraction, which will not affect the value of the fraction, and you will have $Bx = \frac{2BB+2Bs}{4A}$; therefore $Bx+C = \frac{2BB+2Bs}{4A} + \frac{C}{1} = \frac{2BB+2Bs+4AC}{4A} = \frac{BB+2Bs+BB+4AC}{4A} = \frac{BB+2Bs+ss}{4A}$, because $BB+4AC=ss$ by the supposition; therefore $Axx=Bx+C$, since each side is equal to the same quantity $\frac{BB+2Bs+ss}{4A}$.

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CASE 2d.

Let now $x = \frac{B-s}{2A}$, and you will have $xx = \frac{BB-2Bs+ss}{4AA}$,
 and Axx (or the first side of the general equation) $= \frac{BB-2Bs+ss}{4A}$;
 again, $Bx = \frac{BB-Bs}{2A} = \frac{2BB-2Bs}{4A}$; therefore $Bx+C =$
 $\frac{2BB-2Bs+4AC}{4A} = \frac{BB-2Bs+ss}{4A}$; therefore $Axx = Bx+C$,
 because each side is equal to the same quantity $\frac{BB-2Bs+ss}{4A}$.

Various examples of the resolution of affected quadratic equations, both with, and without the general theorem.

EXAMPLE I.

105. Let the equation proposed to be resolved be $6xx = 5x - 1$. This particular equation, as well as all those that follow, may be resolved after the same manner as the general one in art. 103: but as these resolutions are very often attended with fractions very troublesome to the young Analyst, and as these particular equations are nothing else but particular cases of the general one, it follows, that the resolution of these equations must necessarily be included in the resolution of the general one; and consequently, that these equations will be much more easily and readily resolved by referring them to the general one: however for the satisfaction of the learner, I shall resolve some of these equations both with, and without the general theorem; and first I shall resolve the equation proposed by the help of the general theorem thus; in the general equation, art. 103, we have $Axx = Bx + C$; in the particular one already proposed, we have $6xx = 5x - 1$; therefore A in the general equation answers to 6 in the particular one, B answers to 5, and C to -1 ; therefore if the particular equation be referred to the general one, its resolution will be as follows: $A=6$, $B=5$, $C=-1$, $BB=25$, $4AC=-24$; therefore ss , or $BB+4AC$ will be the sum of 25 and $-24=1$; therefore $s=1$, $\frac{B+s}{2A} = \frac{5+1}{12} = \frac{1}{2}$, $\frac{B-s}{2A} = \frac{5-1}{12} = \frac{1}{3}$; therefore the two roots of this equation $6xx = 5x - 1$ are $\frac{1}{2}$ and $\frac{1}{3}$. The resolution of this equation in numbers without the general theorem, is as follows:
 Equation,

Equation, $6xx = 5x - 1$; therefore $6xx - 5x = -1$, and $xx - \frac{5x}{6} = -\frac{1}{6}$; where $xx - \frac{5x}{6}$ may be considered as the two first members of a square raised from a binomial root; the coefficient of the second term is $-\frac{5}{6}$, it's half $-\frac{5}{12}$, and the square of this $\frac{+25}{12 \times 12}$, which expression I choose to make use of rather than $\frac{+25}{144}$ for a reason that will presently be seen; add now $\frac{25}{12 \times 12}$ to both sides, that is, to one side to compleat the square, and to the other to preserve the equality, and you will have $xx - \frac{5x}{6} + \frac{25}{12 \times 12} = -\frac{1}{6} + \frac{25}{12 \times 12}$; here now it is certain that the fractions $-\frac{1}{6}$ and $\frac{+25}{12 \times 12}$ must be reduced to the same denomination in order to be added together into one sum; but if this be done the common way, it will be impossible to obtain the square root of that sum without a further reduction; therefore to avoid this, I enquire what number the denominator 6 must be multiplied by to make it 12×12 the same with the other denominator, and the answer in this case, as well as in all others of this kind will be very easy; for $2 \times 6 = 12$, and therefore $12 \times 2 \times 6$, or $24 \times 6 = 12 \times 12$; therefore I multiply both the numerator and denominator of the fraction $-\frac{1}{6}$ into 24, and so have $-\frac{24}{12 \times 12}$; and this added to the other fraction $\frac{+25}{12 \times 12}$ gives $\frac{+1}{12 \times 12}$; and now the equation will be $xx - \frac{5x}{6} + \frac{25}{12 \times 12} = \frac{1}{12 \times 12}$; extract the root of both sides, and you will have $x - \frac{5}{12} = \pm \frac{1}{12}$, whence $x = \frac{5 \pm 1}{12}$; but $\frac{5+1}{12} = \frac{1}{2}$, and $\frac{5-1}{12} = \frac{1}{3}$; therefore $x = \frac{1}{2}$, or $\frac{1}{3}$.

This may also be proved synthetically thus: let $x = \frac{1}{2}$, then you will have $xx = \frac{1}{4}$, and $6xx = \frac{6}{4}$ or $1\frac{1}{2}$; again, $5x = \frac{5}{2} = 2\frac{1}{2}$; therefore $5x - 1 = 1\frac{1}{2}$; therefore $6xx = 5x - 1$, since each equals $1\frac{1}{2}$.

Let us now suppose $x = \frac{1}{3}$, and you will have $xx = \frac{1}{9}$, and $6xx = \frac{6}{9}$, or $\frac{2}{3}$; on the other hand you will have $5x = \frac{5}{3}$ or $1\frac{2}{3}$; therefore $5x - 1 = \frac{2}{3}$; therefore $6xx = 5x - 1$: these two fractions therefore will answer the

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condition of the equation; and there are no other numbers beside these, whether whole numbers or fractions, that will do it.

EXAMPLE 2.

Let the equation to be resolved be $24x - 2xx = xx + 45$. Here transposing $-2xx$ we have $3xx + 45 = 24x$, whence $3xx = 24x - 45$; and thus we have reduced the equation proposed to the form of the general one in art. 102; wherefore applying that general equation to this particular one, the resolution, by art. 103 will be as follows: $A=3$, $B=24$, $C=-45$, $BB=576$, $4AC=-540$, $ss=576-540=36$, $s=6$, $\frac{B+s}{2A}=5$, $\frac{B-s}{2A}=3$; therefore $x=5$, or 3 ; and this will further easily appear by substituting 5 or 3 for x in the original equation thus; $x=5$; therefore $24x=120$; $xx=25$; therefore $24x - 2xx=120-50=70$, which is one side of the equation: on the other side we have $xx+45=25+45=70$; therefore $24x - 2xx=xx + 45$. Again, let $x=3$, then we shall have $24x=72$, and $xx=9$, and $24x - 2xx=54$: on the other hand, $xx+45=54$; therefore $24x - 2xx=xx + 45$.

N. B. This last equation when reduced to the form of the general one in art. 102, stood thus; $3xx=24x-45$: but this equation might have been reduced to a more simple one of the same form by dividing the whole by 3 , and then the equation would have stood thus, $xx=8x-15$; in which case we should have had $A=1$, $B=8$, $C=-15$, $BB=64$, $4AC=-60$, $ss=4$, $s=2$, $\frac{B+s}{2A}=5$, $\frac{B-s}{2A}=3$, as before: the solution of the foregoing equation in the common way is this; $xx-8x=-15$; therefore completing the square, $xx-8x+16=1$; therefore extracting the square root, $x-4=\pm 1$ therefore $x=\pm 4\pm 1=5$, or 3 .

EXAMPLE 3.

Let the equation to be resolved be $72x - 2xx + 144 = 3xx - 8x + 444$. Here by transpositions we have $72x + 144 = 5xx - 8x + 444$, and $80x + 144 = 5xx + 444$, and $5xx = 80x - 300$, and $xx = 16x - 60$; which equation being resolved like that in the last example, gives $x=10$, or 6 ; which may also be easily seen by substituting 10 or 6 for x in the original equation.

EXAMPLE 4.

Let the equation to be resolved be $28x - xx = 115$. Here we have $xx + 115 = 28x$, and $xx = 28x - 115$; which equation being resolved

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solved like that in the second example, gives $x=23$, or 5 ; the proof whereof is easy.

EXAMPLE 5.

Let the equation to be resolved be $\frac{120}{x} - 5 = \frac{120}{x+4}$; therefore $120 - 5x = \frac{120x}{x+4}$; therefore $100x - 5xx + 480 = 120x$; therefore $5xx + 120x = 100x + 480$; therefore $5xx = -20x + 480$; therefore (dividing by 5) $xx = -4x + 96$; therefore in this case, $A=1$, $B=-4$, $C=96$, $BB=16$, $4AC=384$, $ss=16+384=400$, $s=20$, $\frac{B+s}{2A} = \frac{-4+20}{2} = 8$, $\frac{B-s}{2A} = \frac{-4-20}{2} = -12$; therefore in this equation, $x=8$, or -12 : the proof is thus; let $x=+8$; then $\frac{120}{x} = 15$, and $\frac{120}{x} - 5 = 10$: again, $x+4=12$, and $\frac{120}{x+4} = 10$; therefore $\frac{120}{x} - 5 = \frac{120}{x+4}$. Again, let $x=-12$, then $\frac{120}{x} = -10$; therefore, $\frac{120}{x} - 5 = -10 - 5 = -15$: on the other hand, $x+4=-12+4=-8$; therefore $\frac{120}{x+4} = \frac{120}{-8} = -15$; therefore $\frac{120}{x} - 5 = \frac{120}{x+4}$. The resolution in the common way is this; $xx = -4x + 96$; therefore $xx + 4x = 96$; therefore $xx + 4x + 4 = 100$; therefore $x+2 = \pm 10$; therefore $x = -2 \pm 10 = +8$, or -12 .

EXAMPLE 6.

Let the equation to be resolved be $2xx + 3x = 65$; therefore $2xx = -3x + 65$; therefore in this case, $A=2$, $B=-3$, $C=65$, $BB=9$, $4AC=520$, $ss=529$, $s=23$, $\frac{B+s}{2A} = \frac{-3+23}{4} = 5$, $\frac{B-s}{2A} = \frac{-3-23}{4} = -6\frac{1}{2}$: therefore in this equation, $x=+5$, or $-6\frac{1}{2}$: that $x=+5$ will easily be seen; and that $x=-6\frac{1}{2}$, or that $-6\frac{1}{2}$ being substituted for x will make $2xx + 3x = 65$, I thus demonstrate: $x = -6\frac{1}{2} = -\frac{13}{2}$; therefore $xx = \frac{+169}{4}$; therefore $2xx =$

$\frac{169}{2}$; and $+3x = +3 \times \frac{-13}{2} = \frac{-39}{2}$; therefore $2xx + 3x = \frac{169-39}{2} = \frac{130}{2} = 65$. The resolution in numbers; $2xx + 3x = 65$; therefore $xx + \frac{3x}{2} + \frac{9}{4 \times 4} = \frac{65}{2} + \frac{9}{4 \times 4} = \frac{520+9}{4 \times 4} = \frac{529}{4 \times 4}$; therefore $x + \frac{3}{4} = \pm \frac{23}{4}$; therefore $x = \frac{-3 \pm 23}{4} = +5$, or $-6\frac{1}{2}$.

EXAMPLE 7.

Let the equation to be resolved be $9xx - x = 140$; therefore $9xx = 1x + 140$. Here $A=9$, $B=1$, $C=140$, $BB=1$, $4AC=5040$, $ss=5041$, $s=71$, $\frac{B+s}{2A}=4$, $\frac{B-s}{2A}=-3\frac{8}{9}$; therefore $x = +4$, or $-3\frac{8}{9}$: the latter case I thus demonstrate; $x = -3\frac{8}{9} = \frac{-35}{9}$; therefore $xx = \frac{+1225}{81}$; therefore $9xx = \frac{1225}{9}$: again, $-1x$, that is, $-1 \times \frac{-35}{9} = \frac{+35}{9}$, therefore $9xx - x = \frac{1225+35}{9} = \frac{1260}{9} = 140$. In numbers thus; $9xx - 1x = 140$; therefore $xx - \frac{1x}{9} = \frac{140}{9}$; therefore $xx - \frac{1x}{9} + \frac{1}{18 \times 18} = \frac{140}{9} + \frac{1}{18 \times 18} = \frac{5040+1}{18 \times 18} = \frac{5041}{18 \times 18}$; extract the root of both sides, that is, of $xx - \frac{1x}{9} + \frac{1}{18 \times 18}$ on one side, and of $\frac{5041}{18 \times 18}$ on the other, and you will have $x - \frac{1}{18} = \pm \frac{71}{18}$; whence $x = +4$, or $-3\frac{8}{9}$.

EXAMPLE 8.

Let the equation to be resolved be $\frac{45}{2x+3} + \frac{116}{4x+5} = 7$; therefore $45 + \frac{232x+348}{4x+5} = 14x+21$; therefore $180x+225+232x+348 = 56xx+154x+105$; that is, $412x+573 = 56xx+154x+105$; therefore $258x+573 = 56xx+105$; therefore $56xx = 258x+468$; therefore (dividing by 2) you have $28xx = 129x+234$; which equation being compared with the general one exhibited in art. 103, gives $A=28$,

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$A=28$, $B=129$, $C=234$, $BB=16641$, $4AC=26208$, $ss=42849$,

$s=207$, $\frac{B+s}{2A}=6$, $\frac{B-s}{2A}=-1\frac{11}{28}$; therefore in this equation $x=$

$+6$, or $-1\frac{11}{28}$; both which I thus demonstrate: first $x=6$; therefore

$2x+3=15$; therefore $\frac{45}{2x+3}=3$; moreover, $4x+5=29$; there-

fore $\frac{116}{4x+5}=4$; therefore $\frac{45}{2x+3}+\frac{116}{4x+5}=3+4=7$: second-

ly, $x=-1\frac{11}{28}=-\frac{39}{28}$; therefore $2x=-\frac{39}{14}$; therefore $2x+3=$

$-\frac{39}{14}+\frac{3}{1}=\frac{3}{14}$; therefore $\frac{45}{2x+3}$ is the quotient of $\frac{45}{1}$ divided by

$\frac{3}{14}$; but this quotient, according to the rules of fractional division, is

$\frac{630}{3}=210$; therefore $\frac{45}{2x+3}=210$: again, $4x=-\frac{39}{7}$; therefore

$4x+5=-\frac{39}{7}+\frac{5}{1}=-\frac{4}{7}$; therefore $\frac{116}{4x+5}$ is the quotient of $\frac{116}{1}$

divided by $-\frac{4}{7}$; but this quotient is $-\frac{812}{4}$, or -203 ; therefore

$\frac{116}{4x+5}=-203$; therefore $\frac{45}{2x+3}+\frac{116}{4x+5}=210-203=7$.

The resolution of this equation in the common way is as follows; $56xx$
 $-258x=468$; therefore $xx-\frac{258x}{56}=\frac{468}{56}$: here the coefficient of

the second term is $-\frac{258}{56}$, it's half $-\frac{129}{56}$, and the square of this

$\frac{16641}{56 \times 56}$; add this square to both sides, and you will have $xx-\frac{258x}{56}+$

$\frac{16641}{56 \times 56}=\frac{468}{56}+\frac{16641}{56 \times 56}=\frac{26208+16641}{56 \times 56}=\frac{42849}{56 \times 56}$; extract the square

root of both sides, that is, of $xx-\frac{258x}{56}+\frac{16641}{56 \times 56}$ on one side, and of

$\frac{42849}{56 \times 56}$ on the other, and you will have $x-\frac{129}{56}=\pm\frac{207}{56}$; whence x

$=+6$, or $-1\frac{11}{28}$.

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EXAMPLE 9.

Let the equation be $15x - xx = 56$; then this equation being resolved by the general theorem gives $x = 8$, or 7 ; and in the common way it is thus resolved; $15x - xx = 56$; change all the signs to make xx affirmative, and you will have $xx - 15x = -56$; whence $xx - 15x + \frac{225}{4} = -56 + \frac{225}{4} = \frac{1}{4}$; therefore $x - \frac{15}{2} = \pm \frac{1}{2}$, and $x = 8$, or 7 : but what I chiefly intend by this example is to shew, that in resolving a quadratic equation by the general theorem there is no necessity of making any transposition to exhibit xx affirmative when it would otherwise have been negative; as for instance, in the equation here proposed we had $15x - xx = 56$; transpose $15x$, and you will have $-xx$, that is, $-1xx = -15x + 56$; let this equation be referred to the general one in art. 102, and resolved by the general theorem in art. 103, and you will have $A = -1$, $B = -15$, $C = 56$, $BB = 225$, $4AC = -224$, $ss = 1$, $s = 1$, $\frac{B+s}{2A} = \frac{-15+1}{-2} = \frac{-14}{-2} = +7$, $\frac{B-s}{2A} = \frac{-15-1}{-2} = +8$.

How the learner is to proceed when the roots of a quadratic equation are inexpressible.

106. As there are but few square numbers in comparison of the rest, and as all quadratic equations are resolved by extracting the square root, it follows, that there are but few quadratic equations capable of an exact numeral solution in comparison of those that are not: but as the square root may be extracted to any degree of exactness we please, the resolution of a quadratic equation, which depends upon it, may also be performed to any degree of accuracy whatever; as will appear by the following example.

EXAMPLE 10.

Let the equation be $xx - 4x + 1 = 0$, or $xx = 4x - 1$. Here $A = 1$, $B = 4$, $C = -1$, $BB = 16$, $4AC = -4$, $ss = 12$, $s = \sqrt{12}$, $\frac{B+s}{2A} = \frac{4+\sqrt{12}}{2}$, and $\frac{B-s}{2A} = \frac{4-\sqrt{12}}{2}$; therefore $x = \frac{4+\sqrt{12}}{2}$, or $\frac{4-\sqrt{12}}{2}$; but let us enquire in the next place, whether these two fractions

fractions are not capable of being reduced to more simple terms; first then it is plain that $\frac{4}{2} = 2$, and I say further that $\frac{\sqrt{12}}{2} = \sqrt{3}$; for 12

$= 3 \times 4$; therefore $\sqrt{12} = \sqrt{3} \times \sqrt{4} = \sqrt{3} \times 2$; therefore $\frac{\sqrt{12}}{2} = \sqrt{3}$; whence it follows, that $x = 2 + \sqrt{3}$, or $2 - \sqrt{3}$; but $\sqrt{3}$ extracted to three decimal places gives 1.732: therefore $2 + \sqrt{3} = 3.732$, and $2 - \sqrt{3} = .268$; therefore $x =$ (nearly) 3.732, or .268, as will be further evident from the proof following: first $x = 3.732$; therefore $xx = 13.927824$; and $4x = 14.928$; therefore $4x - xx = 1.000176$; therefore $xx - 4x = -1.000176$; therefore $xx - 4x + 1 = -.000176 = 0$ very nearly: secondly, let $x = .268$ and you will have $xx = .071824$ and $4x = 1.072$, and $4x - xx = 1.000176$; therefore $xx - 4x = -1.000176$; therefore $xx - 4x + 1 = -.000176 = 0$ very nearly; therefore in both cases, the condition of the equation is answered to as many figures or cyphers, as is equal to the number of decimal places to which the square root of 3 was extracted.

It may seem to some perhaps a paradox to assert, that though the two surd values of the unknown quantity found in this and the like cases, are not to be expressed in numbers, yet they may be demonstrated to be just: Thus I shall demonstrate, that if either of the two values of x found in the last case, to wit, $2 + \sqrt{3}$, or $2 - \sqrt{3}$, be substituted for x , we shall have this equation $xx - 4x + 1 = 0$, which was the equation there proposed: in order to this, make $\sqrt{3} = s$; and first, let $x = 2 + \sqrt{3}$, or $2 + s$; and we shall have $xx = 4 + 4s + ss$, and $-4x = -8 - 4s$; and $xx - 4x = 4 + 4s + ss - 8 - 4s = ss - 4$; but if $s = \sqrt{3}$, $ss = 3$, and $ss - 4 = -1$; therefore, $xx - 4x = -1$, and $xx - 4x + 1 = 0$: secondly, let $x = 2 - \sqrt{3}$, or $2 - s$, and we shall have $xx = 4 - 4s + ss$, and $-4x = -8 + 4s$, and $xx - 4x = ss - 4 = -1$, as before; whence $xx - 4x + 1 = 0$.

Of impossible roots in a quadratic equation, and whence they arise.

107. The roots of quadratic equations are not only very often inexpressible, but sometimes even impossible, as will appear by the following example.

EXAMPLE II.

Let the equation be $xx - 4x + 6 = 0$, or $xx = 4x - 6$. Here $A = 1$, $B = 4$, $C = -6$, $BB = 16$, $4AC = -24$, $ss = -8$, $s = \sqrt{-8}$, $\frac{B+s}{2A} = \frac{4+\sqrt{-8}}{2}$, $\frac{B-s}{2A} = \frac{4-\sqrt{-8}}{2}$; but $\frac{4}{2} = 2$, and

and $-8 = -2 \times +4$; therefore $\sqrt{-8} = \sqrt{-2} \times \sqrt{+4} = \sqrt{-2} \times 2$; therefore $\frac{\sqrt{-8}}{2} = \sqrt{-2}$; therefore in this equation, $x = 2 + \sqrt{-2}$, or $2 - \sqrt{-2}$; but as no quantity whatever, either affirmative or negative, being multiplied into itself will produce a negative, it follows, that $\sqrt{-2}$ is not only an inexpressible quantity, but also an impossible one; and consequently, that the two values of x in this equation $2 + \sqrt{-2}$ and $2 - \sqrt{-2}$ will both be impossible.

N. B. Though the roots of this last equation be impossible in their own natures, yet they may be abstractedly demonstrated to be just, as in the last article, by making $s = \sqrt{-2}$, and consequently $ss = -2$.

From what has been said concerning impossible roots, it appears that one root of a quadratic equation can never be impossible alone, but that they must either be both possible or both impossible: for it appears from the resolution of the last equation, that the impossibility of the roots flows from the impossibility of the quantity s , or of the square root of ss when it is negative; now when s is possible, both the roots of the equation $\frac{B+s}{2A}$ and $\frac{B-s}{2A}$ will be possible; on the other hand, when s is impossible, both the roots must necessarily be impossible.

Since the possibility or impossibility of the two roots of a quadratic equation depends upon the quantity ss being affirmative or negative, it follows, that when ss and consequently s equals nothing; the roots will be in the limit between possible and impossible: now if $s = 0$, we shall have $\frac{B+s}{2A} = \frac{B}{2A}$, and $\frac{B-s}{2A} = \frac{B}{2A}$; therefore the two unequal roots of a quadratic equation grow nearer and nearer to a state of equality as they grow nearer and nearer to a state of impossibility, but do not come to be equal till they come to the limit between possibility and impossibility.

How to find the sum and product of the two roots of a quadratic equation without resolving it: also how to generate a quadratic equation that shall have any two given numbers whatever for it's roots.

108. In a quadratic equation of this general form, to wit, $Axx = Bx + C$, the sum of the roots will always be $\frac{B}{A}$, and the product of their multi-

THE
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IN TEN BOOKS:

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VOLUME THE FIRST,
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To which are prefixed

- I. The LIFE and CHARACTER of the AUTHOR.
II. His PALPABLE ARITHMETIC Decyphered.
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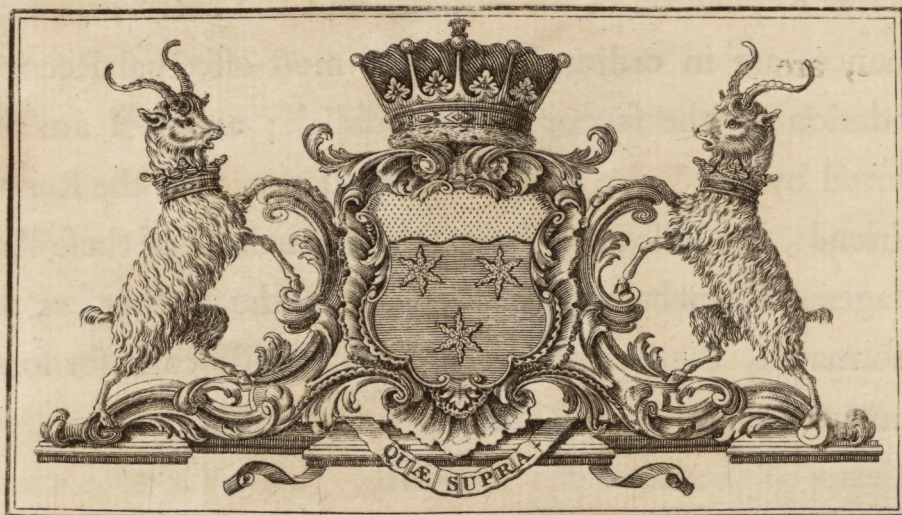
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I. THE LIFE AND CHARACTER OF THE AUTHOR.
II. THE PARABOLIC AND HYPERBOLIC FUNCTIONS.





To the Right Honourable

JOHN Earl of RADNOR,
Viscount BODMYN,
Baron ROBARTES of TRURO, &c.

MY LORD,

THE Relation I bore to the Author of the following Work, gives me a Right to interest myself in the Success of it. To recommend it to the Publick with all possible Advantage, is the least I owe to the Memory of the best of Fathers.

DEDICATION.

THE Reputation of an Author, and the Merit of a Performance, are in ordinary Cases the most effectual Recommendation to the favour of the Publick; and if I am not deceived by the Voice of Fame, nor flattered by the Report of Friends, the following Treatise wants neither of these Advantages. But whatever be the Case of the Author, or the Performance, the Subject itself, I am afraid, calls for some Countenance and Protection.

THE *Belles Lettres*, My LORD, lie open to all Mankind; every one that is capable of reading any thing, either has, or pretends to have some Relish for them. But Science lies more concealed, and more out of the reach of vulgar Apprehensions: Some have not Capacities or a just Taste for Mathematical Truths; and of those that have, few have Resolution enough to pursue them with a proper Industry: and be their Use and Importance ever so great, they will hardly emerge under such Disadvantages without some powerful Encouragement.

THESE, My LORD, are Difficulties arising from the nature of the Subject; to remove which, and to inspire the Youth with a generous Emulation for so noble a Science, would

DEDICATION.

would be doing Service to the Publick, as well as Justice to the Author.

THE Province of Exhortation and Instruction I leave to others, and content myself with the only Method that lies within my reach, of ushering this Work into the World under the Authority and Protection of some great Name; one whose Birth and Quality may give a Dignity to Science, and by being a Master of it himself, has the juster Title to patronize it in others; who does not measure the worth of a Performance by the fulsome Incense of the Dedicator, but penetrates into the Work itself, and learns from thence the real Abilities of the Writer.

IT is for these Reasons, My LORD, that I presume to lay this Treatise at Your Feet. Suffer me to borrow Your LORDSHIP'S Name to throw a Lustre upon my Subject, to attract the Attention of the Lazy and the Prejudiced; and to invite them by so great an Example to an Acquaintance with a Science, whose Use is almost as extensive as Truth itself, and whose Invention does Honour to Human Reason.

PUTTING this Treatise under Your LORDSHIP'S Protection I look upon to be a Part of the Author's Will, and
which

DEDICATION.

which it is incumbent upon me to execute. Had He lived to give it to the Publick, I am confident he would have sought no other Patron. Your LORDSHIP was his first Acquaintance in the University and his first Scholar, and, (what he ever remembered with the sincerest Gratitude) his most generous Benefactor. A great Part of the Work was drawn up originally for Your LORDSHIP's Use, and was for many Years in your sole Possession. And surely, My LORD, there is something of Justice in granting Protection to a Book, which first opened to You the Beauties of Mathematical Knowledge, and to which You are in some measure indebted for that excellent Taſt, that ſo eminently diſtinguiſhes Your LORDSHIP.

THE Author was honoured with the Friendſhip of moſt Perſons of Quality and Condition that ſtudied in the University, though it ended for the moſt Part with their Reſidence there. But Your LORDSHIP raiſes Friendſhips upon more laſting Principles; the Moment a Man of Merit knows You, he becomes your Friend, and when he is once ſo, he can never be otherwiſe. The Author had a Heart extremely ſenſible to true Worth; it received a very ſtrong and durable Impreſſion from the uncommon Share of it he obſerved in Your LORDSHIP; his Eſteem purſued You into the
World,

D E D I C A T I O N.

World, and continued undiminished 'till he left it; he loved and honoured You sincerely and passionately when living, and mentioned You with peculiar Tendernefs and Affection in his dying Moments.

I CANNOT recal with Indifference a Scene in which I bore fo confiderable a Part, though my Youth rendered me incapable of taking my juft Share of it. He was fnatched away at a time when I was unable to make a true Eftimate of the Lofs; when I had only experienced the Tendernefs of the Parent, without enjoying the more folid Benefit of the Friend and the Inftuctor.

BUT I will fhut up a Subject, which can give Your LORDSHIP no Pleafure, and which muft give me the tendereft Concern. It is an Event that had been truly fatal to me, had not Your LORDSHIP's Goodnefs prevented it. This has taught me to bear a Misfortune which is the common Lot of all Men; and 'tis from this I am daily learning (as far as the Ties of Nature and filial Piety will permit) to forget it. Suffer me, My LORD, to boaft (for there is something of Vanity as well as Gratitude in it) that in Your fingular Generofity and Indulgence I have found the fubftantial Comfort of the Friend and the Parent. It is with Pride and
Pleasure

DEDICATION.

Pleasure that I proclaim to the World my Obligations to You,
and it is with the warmest Gratitude, and the most perfect
Regard that I profess myself,

My LORD,

Your LORDSHIP'S

most obliged and

devoted Servant

JOHN SAUNDERSON.

MEMOIRS
OF THE
LIFE and CHARACTER
OF

Dr. *Nicholas Saunderson*,

Late *Lucasian* Professor of the Mathematics
in the University of CAMBRIDGE.

THE following Treatise was entirely finished, before the Public was engaged in the Subscription; but the Author dying during the Impression of the Work, it is believed some short Account of a Person so remarkable and singular will be no unacceptable Entertainment to the Reader. Therefore the following Gentlemen, Dr. THOMAS NETTLETON of *Halifax*, Dr. RICHARD WILKES of *Woolverhampton*, the Rev. Mr. JOHN BOLDERO late Fellow of *Christ's* College, the Rev. Mr. GERVAS HOLMES late Fellow of *Emmanuel* College, the Rev. Mr. GRANVILE WHEELER, and Dr. RICHARD DAVIES of *Shrewsbury* late Fellow of *Queen's* College, who were the intimate Friends of the Deceased, in the different Parts of his Life from his Youth to the time of his Death, have communicated the following Particulars.

c

Mr.

Mr. NICHOLAS SAUNDERSON was born in *January* 1682; at *Thurlston* near *Penniston* in *Yorkshire*. His Father, besides a small Estate, had a Place in the Excise, which he enjoyed above Forty Years with good Repute. His eldest Son, of whom we are speaking, when Twelve Months old, was deprived by the Small Pox, not only of his Sight, but his Eyes also, for they came away in Abscess. A Sense so little enjoyed was soon forgot; he retained no more Idea of Light and Colours than if he had been born Blind.

HE was sent early to the Free School at *Penniston*, and under the Instruction of Mr. STANFORTH, laid the Foundation of that Knowledge of the Greek and Roman Languages, which he afterwards improved so far by his own Application to the Classic Authors, as to hear the Works of *Euclid*, *Archimedes* and *Diophantus* read in their original Greek. *Virgil* and *Horace* were his Favourites among the Roman Poets: His Memory was well stored with their most beautiful Passages, and he would frequently in Conversation quote them with great Propriety. He was well versed in the Writings of *Tully*, and dictated Latin in a familiar and elegant Style. He afterwards acquired a competent knowledge of the French Tongue.

As soon as he had gone through the Business of the Grammar School, his Father, whose Occupation led him to be conversant in Numbers, began to instruct him in the common Rules of Arithmetic. Here it was his Genius first appeared; he soon became able to work the common Questions, to make long Calculations by the strength of his Memory, and to form new Rules to himself for the more ready solving of such Problems, as are often proposed to Learners, more with a design to perplex than instruct: so that in all Difficulties, his Schoolfellows generally applied to him instead of their Master.

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OF PROFESSOR SAUNDERSON. ii

AT the Age of Eighteen he was introduced to the acquaintance of RICHARD WEST of *Underbank* Esq; a Gentleman of Fortune and a Lover of the Mathematics: who observing Mr. SAUNDERSON's uncommon Capacity, took the Pains to instruct him in the Principles of Algebra and Geometry, and gave him every Encouragement in his Power to the Prosecution of these Studies: foreseeing of what Advantage to Letters so great a Genius might be. Soon after, he grew acquainted with Dr. NETTLETON; and it was to the great Pleasure these Gentlemen took in assisting and improving him in his Studies, that our Author owed his first Institution in the Mathematic Sciences. They furnished him with Books, and often read and expounded them to him: but he soon surpassed his Masters, and became fitter to teach, than learn any thing from them.

OUR Author's Passion for Learning grew with him, and his Father, willing to encourage this laudable Disposition, sent him to a private Academy at *Attercliff* near *Sheffield*. Logic and Metaphysics made up the principal Learning of this School. The former being chiefly the Art of Disputing in Mood and Figure, a dry Study, much conversant in Words, the latter dealing in such abstract Ideas as have not the Objects of Sense for their Foundation, were neither of them agreeable to the Genius of our Author; he therefore made but a short Stay here for Instruction.

AFTER he left this Place, he remained some time in the Country, prosecuting his Studies in his own way, without any Guide or Assistant: indeed he needed no other than a good Author, and some Person that could read it to him: by the Strength of his own Genius he could easily master any Difficulty that occurred therein. His Education had hitherto been carried on at the Expence of his Father, who having a numerous Family, grew uneasy under the Burden. His

Friends therefore began to think of fixing him in some way of Business, by which he might support himself. His Inclination led him strongly to the University of CAMBRIDGE, where he expected to meet with the best Opportunities of Improvement in his favourite Studies. But the great Expence of Education, the length of Time he must continue there to obtain his Degrees, and be duly qualified for any of the Liberal Professions, were Difficulties not to be surmounted. At last it was resolved he should try his Fortune there, but in a way very uncommon; not as a Scholar but a Master: for his Friends, observing the extraordinary Proficiency which he had already made in Mathematical Learning, and a peculiar Felicity of Expression in conveying his Ideas to others, were sanguine in their Hopes, that he might teach the Mathematics with Credit and Advantage, even in the University. Or if this Design should miscarry, they promised themselves Success in opening a School for him in *London*.

Accordingly in the Year 1707, being now Twenty-five Years of Age, he was brought to CAMBRIDGE by Mr. JOSHUA DUNN, then a Fellow-Commoner of *Christ's* College, where he resided with his Friend, but was not admitted a Member of the College. The Society were extremely pleased with so unusual a Guest, allotted him a Chamber, the use of their Library, and indulged him in every Privilege that could be of Advantage to him. But many Difficulties obstructed his Design: He was placed here without Friends, without Fortune, a Youth untaught himself, to be a Teacher of Philosophy in an University where it then reigned in the greatest Perfection. Mr. WHISTON was at this time in the Mathematical Professor's Chair, and read Lectures in the manner proposed by Mr. SAUNDERSON; so that an Attempt of this kind looked like an Encroachment on the Privileges of his Office. But as a good-natured Man and an Encourager of Learning

OF PROFESSOR SAUNDERSON. v

Learning, he readily consented to the Application of Friends made in behalf of so extraordinary a Person. Mr. DUNN had been very assiduous in making known his Character; his Fame in a few Months had filled the University, so that Men of Learning and Curiosity grew ambitious and fond of his Acquaintance. His Lecture, as soon as opened, was attended by many from several of the Colleges, and in some time was so crowded, that he could hardly divide the Day among all who were desirous of his Instruction: few, whose Inclination led them to the severer Studies, but eagerly embraced the Opportunity of laying a Foundation in Mathematics and Philosophy under so great a Master.

Sir ISAAC NEWTON had left *Cambridge* several Years before Mr. SAUNDERSON came thither. His *Principia Mathematica* had been long since published, but were at first overlooked, and not sufficiently understood by the World. It was one Design of this Treatise to demolish the *Vortices* and other romantic Chimæras of DES CARTES: and now the Learned began to be sensible how much the Author had done towards a Reformation in Philosophy; which before had been founded upon very erroneous Principles, and *Hypotheses* feigned in the Closet, without one Experiment to shew their Reality in Nature. Sir ISAAC, ever studious of brevity, had drawn up his Demonstrations in the concise manner possible; leaving the Mathematical Reader to furnish himself with every thing before known, and often to take large Steps alone. His Treatise of *Optics* and his *Arithmetica Universalis* were both written in the same masterly Style, and each contain great and peculiar Discoveries. Mr. SAUNDERSON made these several Pieces the Foundation of his Lecture; they afforded a noble Field to display his Genius in; and the Public Schools of the University did sufficiently testify his Success. For those wonderful *Phænomena* of Nature, whose So-
lution

lution was before attained with Difficulty by the best Mathematicians, became the *Theses*, which the Youth of three or four Years standing defended in their Disputations for their first Degree in Arts. We every Year heard the Theory of the Tydes, the *Phænomena* of the Rainbow, the Motions of the whole Planetary System as upheld by Gravity, very well defended by such as had profited by his Lectures.

IT will be matter of surprise to many that our Author should read Lectures in Optics, discourse on the Nature of Light and Colours, explain the Theory of Vision, the Effect of Glasses, the *Phænomena* of the Rainbow, and other Objects of Sight: but if we consider that this Science is altogether to be explained by Lines, and subject to the Rules of Geometry, it will be easy to conceive that he might be a Master of these Subjects.

AS Mr. SAUNDERSON was instructing the University Youth in the Principles of *Newtonian* Philosophy, it was not long before he became acquainted with the incomparable Author, and enjoy'd his frequent Conversation concerning the more difficult Parts of his Works. Dr. HALLEY, Mr. DE MOIVRE and many of the most noted Mathematicians in *London* highly esteemed his Friendship, and in deference to his strong Reason and Judgment, frequently consulted him concerning their Writings and Designs.

UPON the removal of Mr. WHISTON from his Professorship, Mr. SAUNDERSON's Mathematical Merit was universally allowed so much superior to that of any Competitor in the University, that an extraordinary Step was taken in his Favour, to qualify him with a Degree, which the Statutes require. Upon Application made by the Heads of Colleges to the Duke of SOMERSET their Chancellor, together with the Intercession of the Honourable FRANCIS ROBARTES Esq; a Mandate was readily granted by the QUEEN, for conferring on him the Degree

OF PROFESSOR SAUNDERSON. vii

gree of Master of Arts. Upon which he was chosen *Lucasian* Professor of the Mathematics in *November 1711*. During this whole Transaction Sir ISAAC NEWTON interested himself very much in his Favour.

AT this time the ingenious Mr. ROGER COTES filled the *Plumian* Chair of Astronomy and Experimental Philosophy; a Man of great sweetness of Temper, and engaged to our Author in the strictest Friendship; of the same Age, of the same Genius and Inclination to the Mathematics, both approved and recommended to Professorships by Sir ISAAC NEWTON. No University could ever at one time boast of two so capable and so disposed to promote the Study of Philosophy among her Pupils. Had they lived to more mature Ages, mutually assisting and inspiring each other in the pursuit of Knowledge, what Glory might have accrued to our University, what Advancement to Science from their united Labours! But Mr. COTES was hurried away by a Fever in the Flower of his Age, having only time to compose a few Pieces, as Specimens of his extraordinary Capacity, but of great value to the Learned. And our Author's Life, though longer, was so devoted to Lectures, that he now leaves to Posterity as few Monuments of his Abilities.

OUR Author's first Performance after he was seated in the Chair, was an Inauguration Speech, made in very elegant Latin, and a Style truly *Ciceronian*: it was delivered with such just Elocution, and in a manner so graceful, as to gain him the universal Applause of his Audience. In it he first returned his Thanks to Her MAJESTY for the Royal Mandate, to the Chancellor for his ready Application to the Queen, and to the Electors and the rest of his Friends for their good Opinion of his Abilities and Mathematical Knowledge. To these he added a long and noble Encomium on the

the Mathematics, shewing the Excellence and Advantage of this above every other Method of Reasoning.

FROM this time he applied himself closely to the reading of Lectures, and gave up his whole time to his Pupils: so that his Friends soon lost all the Pleasure of his Conversation. He continued among the Gentlemen of *Christ's College* till the Year 1723, when he took a House in *Cambridge*, and soon after married a Daughter of the Rev. Mr. WILLIAM DICKONS late Rector of *Boxworth* in the County of *Cambridge*; by whom he had a Son and a Daughter, both now living.

IN the Year 1728, when His present Majesty King GEORGE the Second honoured the University of *Cambridge* with a Royal Visit, He was pleased to signify his desire of seeing so remarkable a Person. Accordingly our Professor attended upon His Majesty in the Senate House, and was there created Doctor of Laws by his Royal Favour.

Dr. SAUNDERSON was naturally of a strong, healthy Constitution; but being too sedentary, and constantly confining himself to his House, he became at length a Valetudinarian of a very Scorbutic Habit. For some Years he frequently complained of a Numbness in his Limbs, which in the Spring of the Year 1739 ended in a Mortification in his Foot. His Blood was in so ill a State that no Art or Medicines were able to stop its Progress. He died the 19th of *April* 1739; in the Fifty-seventh Year of his Age, and lies buried according to his last Request in the Chancel at *Boxworth*.

AFTER his Life, it may be expected that some Account shall be given of his Character likewise; but I am at a loss for Colours, strong enough to paint a Character so bright and uncommon, and where to place it for View, in the truest Point of Light. A blind Man moving in the Sphere of a Mathema-

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Mathematician seems a *Phænomenon* difficult to be accounted for; and has excited the Admiration of every Age in which it has appeared. *Tully* mentions it as a thing scarce credible in his own Master in Philosophy *Diodotus*^a, “that he exercised himself therein with more assiduity, after he became blind: and what he thought next to impossible to be done without Sight, that he professed Geometry, describing his Diagrams so expressly to his Scholars, that they could draw every Line in its proper Direction.” *St. Jerom* relates a more remarkable Instance in *Didymus*^b of *Alexandria*, who “though blind from his Infancy, and therefore ignorant of the very Letters, appeared so great a Miracle to the World, as not only to learn Logic, but Geometry also to perfection, which seems the most of any thing to require the help of Sight.” The Character of *Didymus* is celebrated, among other Historians, by *Cassiodorus*; who makes mention also of one *Eusebius*^c an *Asiatic*, who according to his own Account of himself, “had been blind from five Years old, and yet had treasured up in his Mind all kinds of Learning, and explained them likewise with the greatest clearness to others.” And *Trithemius* gives a like Instance in one “*Nicaise*^d of *Mechlin*, who though blind from

^a Cic. Tusc. Disp. V, 39. *Diodotus Stoicus, cæcus multos annos, nostræ domi vixit: is verò, quod credibile vix esset, cùm in Philosophia multò etiam magis assiduè quam antea versaretur, — tum quod sine oculis fieri posse vix videtur, Geometriæ munus tuebatur, præcipiens discipulis, unde, quo, quamque lineam scriberent.*

^b Hieronymus de viris Illust. Cap. cix. *Didymus Alexandrinus captus a parva ætate oculis, & ob id elementorum quoque ignarus, tantum miraculum sui omnibus præbuit, ut Dialecticam quoque & Geometriam, quæ vel maxime visu indiget, usque ad perfectum didicerit.*

^c Cassiodorus de Inst. Div. Liter. cap. 5. tradit de partibus *Asiæ* quendam ad nos venisse *Eusebium* nomine, qui se infantem quinque annorum sic cæcatum esse narrabat, ut sinistrum ejus oculum fuisse excavatum orbis profundissimus indicaret: dexter verò globus vitreo colore confusus sine videndi gratia infructuosus nixibusolvebatur. Hic — disciplinas omnes & animo retinebat, & expositione planissima lucidabat.

^d Trithemius de Scriptoribus Eccles. N. DCCCLXXVI, *Nicasus de Voerda, Mechliniensis, — captus à tertio ætatis suæ anno oculis, — secundum nostræ ætate Didymum Alexandrinum exhibuit, dum in omni doctrina & scientia, tam divina quam humana eruditissimus evasit. Nam in gymnasio Colonienfi — jura publicè docuit, libros utriusque juris, quos nunquam vidit, auditu didicit, tenuit mente, apertè recitavit.*

“ the third Year of his Age, yet, like another *Didymus*, became so great a Master of all Learning and Knowledge, “ divine and human, that in the University of *Cologne*, he “ publicly taught the Canon and Civil Law, openly reciting “ Books which he had never seen, but had learnt by only “ hearing them read to him.” Beside these I have heard mention made of an *Hollander*, if not some others, who notwithstanding their Blindness have excelled in Mathematical Learning.

It is remarkable of the few, who have laboured under this Defect, and the still fewer, who had Genius enough to surmount the Difficulties attending it, that so many should be found to excel in Learning, and particularly in the Mathematics, as the two first above mentioned *certainly* did, and *probably* the others also. But if we consider that the Ideas of extended Quantity, which are the chief Objects of Mathematics, may as well be acquired from the Sense of Feeling as that of Sight; that a fixed and steady Attention is the principal Qualification for this Study, and that the Blind are by necessity more abstracted than others, we shall perhaps find Reason to think there is no other Branch of Science more adapted to their Circumstances. It is said of *Democritus* that he put out his Eyes, to enable him to think the more intensely; “ imagining, says *Tully*^e, the Acuteness “ of the Mind was taken off by the Sight of the Eye.” And it was an Observation frequently made by our Professor, that Diagrams which are intended only as helps to the Imagination, are often the means of misleading the Judgment. It is certain, however useful they may be to the Learner, yet the Inventer must in all Cases proceed without them. The Scheme must be erected in his Imagination, in Circumstances as ge-

^e Cic. Tusc. Disp. V, 39. *Democritus* — *impediri etiam animi aciem aspectu oculorum arbitratur.*

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neral as the Propofition, fuch as cannot be delineated upon Paper. And I am confident, that any one who is defirous of more than a general Knowledge of thefe Things, who would invent and improve upon what is to be learned from Books, will find his Mind greatly affifted and enlarged, by accuftoming himfelf to think and reafon in the Circumftances of a blind Man. But a Perfon who has the Misfortune to be fuch, and is deprived of all the Pleafures of Sight, will more frequently and more clofely retire into himfelf, and finding few other Amusements but in the purfuit of Truth, will be more likely to excel in thefe abftraft Sciences.

THE fame Circumftance may poffibly contribute fomething towards raifing the Genius beyond its natural Pitch, in fome other Arts, particularly Mufic and Poetry. The Poet indeed muft firft have his Imagination filled with all the beautiful Variety of Images in Art and Nature, which the Sight only can fupply: if he then be deprived of that Senfe,

*So much the rather may Celeftial Light
Shine inward, and the Mind through all her Powers
Irradiate: —*

as our blind Poet expreffes it. Accordingly in the Catalogue of Epic (the fublimeft kind of) Poets, we find two *blind Bards*, furpaffing all that any Age or any Nation have produced in the Flights of Fancy. And I cannot but wonder the very ingenious *Enquirer* into the Life and Writings of *Homer*, who endeavours to account for the great Genius, from a Concurrence of natural Caufes, fhould take no notice of that Circumftance, which was fo peculiar to his Poet.

IT was by the Senfe of Feeling our Author acquired moft of his Ideas at firft: and this he enjoyed in great Acutenefs and Perfection, as it commonly happens to the Blind, whether by the kind Gift of Nature, or the Neceffity of Application.

tion. Yet he could not, as some have imagined, (and as Mr. Boyle was made to believe of a blind Man at *Maestricht*) distinguish Colours by that Sense; and having made repeated Tryals himself, he used to say, it was pretending to Impossibilities. But he could with great Nicety and Exactness discern the least Difference of Rough and Smooth in a Surface, or the least Defect of Polish. Thus he distinguished in a Set of *Roman* Medals, the genuine from the false, though they had been counterfeited with such Exactness as to deceive a *Connoisseur*, who had judged by the Eye. But says the Professor, "I, who had not that Sense to trust to, could easily feel a Roughness in the new-cast, sufficient to distinguish them by." His Sense of Feeling was very accurate in distinguishing the least Variation in the Atmosphere. I have been present with him in a Garden, making Observations on the Sun, when he has taken notice of every Cloud that disturbed our Observation, almost as justly as we could. He could tell when any thing was held near his Face, or when he passed by a Tree at no great Distance, provided the Air was calm, and little or no Wind: these he did by the different Pulse of the Air upon his Face.

I wish I were capable of entertaining the Curious with the many Contrivances he had, to supply his Defect of Sight. He had a Board made with Holes bored at the equal Distance of half an Inch from each other: Pins were fixed in them, and by drawing a Piece of Twine round their Heads, he could more readily delineate all rectilinear Figures used in Geometry, than any Man could with a Pen. He had another Board with Holes made in right Lines for Pins of different Sizes. By the help of these he could calculate, and set down the Sums, Products, or Quotients in Numbers, as exactly as others could by Writing. By the help of an Armillary Sphere, the Schemes in Geometry that lie in different Planes,

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Planes, and the regular Solids cut in Wood, and the Form of several Curves made after the same manner, he was able on these Subjects to convey the clearest Ideas to his Pupils.

A refined Ear is what such are commonly blessed with, who are deprived of their Eyes. Our Professor was perhaps inferior to none in the Excellence of his: he could readily distinguish to the fifth Part of a Note, and by his Performance on the Flute, which he had learned as an Amusement in his younger Years, discovered such a Genius for Music, as would probably have appeared as wonderful, as his Excellence in the Mathematics; had he cultivated that Art with equal Application. By his Quickness in this Sense he not only distinguished Persons, with whom he had ever once conversed, so long as to fix in his Memory the Sound of their Voice, but in some Measure Places also. He could judge of the Size of a Room into which he was introduced, of the Distance he was from the Wall: and if ever he had walked over a Pavement in Courts, Piazzas, &c, which reflected a Sound, and was afterwards conducted thither again, he could exactly tell whereabouts in the Walk he was placed, merely by the Note it sounded.

THE Reader must greatly admire the Strength of his Memory, when assured that he could calculate in his Mind, multiply, divide, extract the square or cube Root to many Places of Figures; could go along with any Calculator in working Algebraical Problems, Infinite Series, &c; and immediately correct the Slips of the Pen, as well in Signs as in Numbers. Those who read to him had frequent Occasions of admiring his great Sagacity and Quickness of Conception; with how much ease he followed any Track of Reasoning, and with what Art he stored up in his Mind such Parts, as would serve him to recollect and ruminare upon the whole. Indeed in the more abstruse Parts of Mathematics, where the
Scheme

Scheme was very intricate and perplexed, they often found it difficult to raise in his Imagination a clear and distinct Perception of it: but that once done, he seldom or never required any farther Assistance; his Mind retained so strongly every Impression that was once rightly made upon it. By the help of these strong Faculties, a clear Imagination, tenacious Memory, and quick Reason, the Books of Mathematics lay ever open to him; he saw the whole in one View, every Dependency in the Chain of Truth. Thus he knew how to found every thing on the most easy Principles, and to compose with the justest Symmetry and Order.

As in the Knowledge of the Mathematics he was exquisite, and equal to any, so in the Address of a Teacher, he was perhaps superior to all. This Quality was conspicuous at his first Appearance in the World, and must have been highly improved by long Use and Experience. He seemed perfectly to know what Difficulties young Minds are apt to be involved in, and how best to obviate or remove them. His Expression was strong and clear; and his Method so just and natural, that no one was at a loss to follow him. He was very happy in all the Arts of facilitating a Demonstration, in forming curious Positions to help the Imagination and obviate the Difficulties of Conception. I dare appeal to the following Sheets to determine, whether the several Propositions, which have passed through the Hands of *Euclid*, *Archimedes*, *Diophantus*, and the greatest Masters both ancient and modern, have not been greatly improved under his, by lowering the Ground-Work, and rendering the Structure more plain, yet more useful and substantial.

HIS Inclination led him to those Parts of the Mathematics, which are not the most abstracted, and end only in Contemplation. A Proposition must have its Uses, in order to engage his Attention. Either the Method of Enquiry must help

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help to form the Mind, and teach new Modes of Reasoning, or the Proposition itself must tend to some Good, to the Improvement of Life or Science. He considered Mathematics as the Key to Philosophy, as the Clue to direct us through the secret Labyrinths of Nature; and thought the Mind was more highly entertained as well as improved in unravelling Her Works, than investigating the most subtle Properties of abstract Quantity.

As to the *Geometric* and *Analytic* Methods of Reasoning, each of which have their Advocates and Favourers among the Mathematicians of the present Age: our Professor, I think, did Justice to both, in allowing each the Advantage on different Occasions, and making use of that which seemed the most proper for the present. The *Geometric* being the most Intuitive, and conveying the strongest and clearest Ideas to the Mind, he allowed preferable, where equally obvious and easy of Application. But as it was often otherwise, the *Analytic* advancing us in Science much faster and farther than we could have gone by all the Methods of the Ancients, and being the very Art and Principle of Invention, He thought the Moderns were greatly assisted by the use of it.

OUR Professor would not be induced by the Desires and Expectations of any, to engage in the War that was lately waged among Mathematicians, with no small Degree of Heat, concerning the *Algorithm or Principles of Fluxions*. Yet he wanted not the greatest Respect for the Memory of Sir ISAAC NEWTON, and thought the whole Doctrine entirely defensible by the strictest Rules of Geometry. He owned indeed that the great Inventer, never expecting to have it canvassed with so much trifling Subtilty and Cavil, had not thought it necessary to be guarded every where by Expressions so cautious as he might have otherwise used: for he wrote only for such sincere Lovers of Truth as himself was. But the general
Aversion

Aversion he had to all Controversial Writings withheld him from appearing in this. However, as he intended another Volume to his *Algebra* on the *Fluxionary* Part, he there designed to be particularly accurate and explicit upon the *Algorithm*; with an indirect View to the Controversy on foot, and to obviate, the best he could, every Difficulty that had been started.

I cannot upon this Occasion pass by the Name of Sir ISAAC NEWTON, without mentioning the profound Veneration paid to it by our Professor. If he had ever differed in Sentiment from any of his Mathematical and Philosophical Writings, upon more mature Consideration, he said, he always found the Mistake to be his own. The more he read his Works, and observed upon Nature, the more Reason he found to admire the Justness and Care, as well as Happiness of Expression of that incomparable Philosopher. He has left some valuable Comments on the *Principia*, which not only explain the more difficult Parts, but often improve upon the Doctrines, and which may in their present State be no unacceptable Present to the Public, though far short of any thing he would himself have published on that Subject.

THERE was scarce any Part of the Mathematics, on which our Professor had not wrote something for the use of his Pupils. But he discovered no Intention of publishing any of his Works, till the Year 1733: when his Friends, alarmed by a violent Fever that had highly threatened his Life, and being unwilling that the Labours of so great a Man should be lost to the World, importuned him to spare some time from his Lectures, (which he then attended seven or eight Hours in a Day, to the great hazard and prejudice of his Health,) and to employ it in finishing some of his Works; which he might leave behind him, as a valuable Legacy both to his Family and the Public. He yielded so far to these Intreaties,



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treaties, as to compose in a very short time the following Work, which he left perfect, and transcribed fair for the Press.

I NEED not say much of the Nature of it, but refer the Reader to the Contents, and to the Book itself. I will only observe, it was chiefly intended for the Instruction of young Beginners, and for the use of those that had such under their Care. The Author has therefore been very explicit and accurate in the exposition of every Part, and taken great care to express his Sentiments in the most simple and easy manner. His Design was not only to complete a Course of Algebra, but also, as far as possible, to promote the Study of Geometry, by removing or explaining all those Difficulties, which by his long Experience in teaching, he found were apt to retard, if not discourage young Students in their Progress through the Elements. And further, as Algebra is in it's own Nature an Art of Reasoning, and may be considered as the Logical Institutes of the Mathematician, the Author has been every where attentive to improve the Mind, and to furnish it with every Method of Reasoning that may be useful in our Researches into Nature. He has often exposed the same Truths to us in several Lights, as we arrive at them by different Methods of Enquiry: since this served to illustrate the Consistency of those Methods. He has also taken every Occasion to observe the Transition of Truth from one Law to another: to observe the Consistency of it's several Laws in the most intricate Cases, where they seem most to thwart and contradict each other, as if Nature were put to her Shifts to preserve that Consistency and Uniformity which is every where the Characteristic of Truth. Such Observations cannot but suggest to the Mind the most simple and natural Ways of discovering Truth, and must therefore be greatly instructive, as well as entertaining, to all who are engaged in these Enquiries.

BUT the learned Mathematician must not wholly form his Judgment of the Author's Capacity from the following Work; which is of a Nature too low for the Exercise of so great a Genius. Yet the Reader will from hence learn to lament, with me, the Loss of a Life so valuable to the learned World, when he was just entering upon Designs, which, had he lived to execute, would have greatly enriched our Treasure of Mathematic Learning, and have proved Monuments to latest Posterity worthy of his Name. His Manuscripts were all left (for the Benefit of his Children) to the Care and Disposal of JOHN ROBARTES of *Twickenham* Esquire, now Earl of RADNOR; from whose Love and Esteem for Letters and learned Men, and particularly the learned Author, the Public may be well assured, that these Remains will be so disposed of, as to be most advantageous to Science, and honourable to their Author.

THE Talents of Dr. SAUNDERSON were not confined to the Study: when he put on the Companion, none supported Conversation with greater Wit and Elegance. His Discourse was so enlivened with frequent Allusions to Objects of Sight, that there appeared no Defect of the blind Man. Nothing was observed of that Disrelish of Humour, nothing of those Absences and Inattention to Discourse, which usually blemish and characterise Persons devoted to these severer Muses. His Judgment on the various Passions and Interests of Mankind was equally acute as on the Subjects of Philosophy. The Force and Spirit of his Expression surprised and fixed the Attention of all that heard him. But above all, the Mathematician's Reverence for Truth shone forth in every Circumstance of Life and Conversation, and added a Lustre to his most shining Qualities. His Sentiments on Men and Opinions, his Praises or Censures, his Friendship or Disregard were expressed without partiality or reserve. This Frankness
of

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of Temper endeared him to all such as were happy in his Acquaintance and Esteem; but raised him up what Enemies he had, and betrayed him to several Animofities, which Men of more Art and Complaisance would have chose to avoid, at the Expence of so scrupulous and so difinterested a Sincerity.

IT would be thought an Omission in these Memoirs of the Life of Dr. SAUNDERSON, if no notice were taken of the manner in which he resigned it. The Reverend Mr. GERVAS HOLMES informed him, that the Mortification gained so much ground, that his best Friends could entertain no hopes of his Recovery. He received this notice of his approaching Death with great Calmness and Serenity; and after a short Silence, resumed Life and Spirits, and talked with as much Composure of Mind as he had ever done in his most sedate Hours of perfect Health. He appointed the Evening of the following Day to receive the Sacrament with Mr. HOLMES; but before that came, he was seized with a Delirium, which continued to his Death.

Dr. SAUNDERSON's
PALPABLE ARITHMETIC
DECYPHER'D.

THE Author of the following Piece, (who is Dr. SAUNDERSON's immediate Successor in the Professorship,) has been prevailed on to let it be inserted here, as an Illustration to some of his Performances; though it was originally designed for another Place.

THAT the learned and ingenious Dr. SAUNDERSON, late *Lucasian* Professor of Mathematics in the University of CAMBRIDGE, notwithstanding the loss of his Sight, was able to make long and intricate Calculations, both Arithmetical and Algebraical, is a Thing as certain as it is wonderful. This appears beyond all Contradiction, not only from his elaborate Treatise of Algebra now published, but from other undoubted Monuments still in being. He had contrived for his own use, a commodious Notation for any large Numbers, which he could express on his *Abacus*, or Calculating Table, and with which he could readily perform any Arithmetical Operations, by the Sense of Feeling only; which therefore may be called his *Palpable Arithmetic*. As I have had an Opportunity, by the favour of Mrs. SAUNDERSON, of viewing and examining several Specimens of this Arithmetic, which by good fortune he had compleated and left behind him, though he has not left the least Hint by which his Method might be discovered; I had the Curiosity to propose to myself

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self the decyphering (as it may be called) of these Specimens, in which I have succeeded to my own Satisfaction. And as others may have the same Curiosity, or as this Method may possibly be of use to other Persons, whose Misfortune may place them in like Circumstances, I shall here attempt to give a succinct but particular Account of it.

HIS Calculating Table was a smooth thin Board, something more than a Foot square, raised upon a small Frame so as to lie hollow; which Board was divided by a great Number of equidistant parallel Lines, and by others as many, at right Angles to the former. The Edges of the Table were distinguished by Notches, at about half an Inch distance from one another, and to each Notch belonged five of the afore-said Parallels; so that every square Inch was divided into an Hundred little Squares. At every Point of Intersection the Board was perforated by small Holes, capable of receiving a Pin; for it was by the help of Pins, stuck up to the Head through these Holes, that he expressed his Numbers. He used two Sorts of Pins, a larger and a smaller Sort; at least their Heads were different, and might easily be distinguished by feeling. Of these Pins he had a large Quantity in two Boxes, with their Points cut off, which always stood ready before him when he calculated. And these were his Instruments, of which we must now see the use.

IN order to this we may first observe, that to every numeral Figure a little Square was appropriated on the Table, consisting of four of the little contiguous Squares above described, and which therefore allowed a small Interval between each Figure; and this numeral Figure was different, according to the different Magnitude or Situation of the one or two Pins, which always composed it. For which Purpose he had settled in his Mind, and strictly observed, the following Analogy or Notation. A great Pin in the Center of the Square (which,

(which, and no other, was always its Place,) was a Cypher, or 0, and therefore I shall call it by that Name. Its chief Office was, to preserve Order and Distance among his Figures and Lines. This Cypher was always present, except only in the Case of an Unit; to express which the great Pin in the Center was changed into a little one. When 2 was to be expressed, the Cypher was restored to its Place, and the little Pin was put just over it. To express 3, the Cypher remained as before, and the little Pin was advanced into the upper Angle on the right Hand. To express 4, the little Pin descended, and immediately followed the Cypher. To express 5, the little Pin descended to the lower Angle on the right Hand. To express 6, the little Pin retreated, till it was just under the Cypher. To express 7, the little Pin retreated into the lower Angle on the left Hand. To express 8, the little Pin ascended, till it was just before the Cypher. Lastly, to express 9, the little Pin ascended into the upper Angle on the left Hand. And thus all the Digits were provided for, by an easy and uniform Notation, which might readily enough be apprehended and distinguished by the Feeling. But to shew these Digits or Figures more distinctly, I shall endeavour to represent them by a Scheme. See *Fig. I.*

AND thus he could write down, as we may say, any proposed Number upon his Table, and by lightly running his Fingers over it, he could at any time readily read it, and know what it signified. The great Pins or Cyphers, which were always placed at the Centers of the little Squares, and most frequently at equal Distances from one another, were a sure Guide to direct him to keep the Line, to ascertain the Limits of every Figure, and to prevent any Ambiguity that might otherwise arise. As three of the erect Parallels were sufficient for a single Figure, so three of the transverse Parallels would suffice for a Line of Figures, and the next three
for

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for another Line, and so on, without any Danger of interfering. And thus it is not hard to conceive, how he might have any Number of Lines of Figures upon his Table at the same time, in a descending order, or how he might derive one Number from another, or in a word, how he might make any Computations required. He could place and displace his Pins, as I have been informed, with incredible Nimbleness and Facility, much to the Pleasure and Surprize of all the Beholders. He could even break off in the middle of a Calculation, and resume it when he pleased, and could presently know the Condition of it, by only drawing his Fingers gently over the Table. There is an obvious Expedient, which, especially in long Calculations, would have made the Process very expeditious, and therefore I question not but he had often recourse to it. And that is, to prepare the Table beforehand, (which any other might have done for him,) by filling every third Hole of every third parallel Line with large Pins, or Cyphers. Then when he intended to calculate, he would have nothing else to do, but to compleat every Figure, by adding a small Pin in its proper Place. Except when an Unit was to be expressed, in which case he must have changed the large Pin into a small one.

THE Specimens of this Arithmetic which I have perused, and reduced to common Numbers, are certain Arithmetical Tables, which he had computed and preserved for his own use; but for what Purposes they were calculated does not easily appear. They seem to have some relation to the Tables of natural Sines, Tangents, and Secants; but their full use I must leave to future Enquiry. They are four Pieces of solid Wood, of the Form of rectangular Parallelepipeds, each about eleven Inches long, five and an half broad, and something above half an Inch thick. The two opposite Faces of every one were divided into little Squares, after the manner
of

tirely deprived of the use of visible Symbols, which to us we find are so absolutely necessary, as that we always acquire the greatest and most valuable Part of our Mathematical Knowledge by their Means. What did he do under this (as it should seem) insuperable Difficulty, in order to satisfy his great Thirst after this kind of Knowledge? why, he had recourse to another of his Senses, which he had in great Perfection, and substituted Feeling in the place of Seeing; by inventing a Sort of Mathematical Symbols, which we may call palpable or tangible Symbols. These he made use of, to convey those Ideas to his Understanding, which were denied entrance through his Eyes. Thus by the assistance of these Symbols, imperfect and inadequate as they needs must be, and by the help of a quick Apprehension and obstinate Perseverance, he succeeded in these Sciences to Admiration. These were the Instruments, as unfit as they may seem, by which he conveyed to his Mind the most abstruse and sublime Mathematical Ideas, and by which he was enabled to deduce from them the most general and useful Conclusions. This we must all confess, if we have any degree of Candour, as well as Eyes and Understanding, to be very extraordinary and surprising. I cannot but represent to my Imagination, during the whole course of his Studies, the Idea of a great Mathematical Genius, struggling under the greatest Disadvantages, and labouring under the severest natural Inabilities and Discouragements. Yet whatever Difficulties he meets in his way, by a happy Sagacity and stubborn Industry, he finds as many Expedients to overcome them all; resolving not to relinquish his pursuit, but steadily to persist, till he is superior to all Obstacles, and till he has gratified the no mean Ambition, of placing himself in the first Class of Mathematicians.

A Second Advertisement.

SINCE the printing of the former Advertisement, I have been favoured with the use of an English Translation of Napeir's Book, entitled A Description of the admirable Table of Logarithmes, perused and approved by the Author himself, and printed in the year 1616. From which Book it appears, that He himself, and not Briggs, was likewise the inventer of the System of Logarithms now in use, as I had before observed. For thus he writes, book 1. chap. 4. sect. 9. pag. 19, 20. "But because the addition and subtraction of these former numbers may seeme somewhat painfull, I intend (if it shall please God) in a second Edition, to set out such Logarithmes as shall make those numbers about written to fall upon decimal numbers, such as 100 000 000, 200 000 000, 300 000 000, &c. which are easie to be added or abated to or from any other number." Another quotation to the same purpose, from the Author's Rabdologia, published in the year 1617, may be seen in Mr. Professor WARD's Lives of the Professors of Gresham College, p. 122.

Amongst Professor SAUNDERSON's Questions for Exercise in the Rule of Three, there are some which belong to the Double Rule of Three, where five numbers are given in order to find a sixth; as in the thirtysecond question, pag. 13. If two men in three days will earn four shillings, how much will five men earn in six days? Where the numbers given are 2, 3, 4, 5, 6; the three first of which are always in the conditional part of the question, and the other two in the remaining part, which moves the question. It is also to be noted, that in all questions belonging to the Double Rule of Three, the numbers are so to be placed, as that the first and the fourth, the second and the fifth, the third and the number sought, may be of the same denomination respectively; as in the question proposed: 2 men, 3 days, 4 shillings; 5 men, 6 days, shillings required.

*These things being premised, the Professor's first rule is this, pag. 14. In all questions of this nature, if the three last numbers be multiplied together, and the product be divided by the product of the two first, the quotient will give the number sought. As in the question proposed, $\frac{4 \times 5 \times 6}{2 \times 3} = 20$, the number sought. Again, in the thirtythird question, If for the carriage of 3 hundred weight 40 miles I must pay 7 shillings and 6 pence, what must I pay for the carriage of 5 hundred weight 60 miles? The terms are 3 hundred weight, 40 miles, $7\frac{1}{2}$ shillings; 5 hundred weight, 60 miles. *Answ.* $\frac{7\frac{1}{2} \times 5 \times 60}{3 \times 40} = 18\frac{1}{2}$, that is, 18 shillings and 9 pence.*

There is another case, wherein inverse proportion is partly concerned, as in the thirtyninth question, pag. 16. If two acres of land will maintain three horses four days, how long will five acres maintain six horses? Here it is plain, that inverse proportion is concerned: for the same quantity of land will maintain six horses but half the time that it would maintain three. Wherefore for resolving questions of this nature, the Professor proposes a second rule, pag. 17, 18. In all questions belonging to the Double Rule of Three inverse, where the numbers are supposed to be ordered as in the Double Rule of Three direct, if the three middle numbers be multiplied together, and the product be divided by the product of the two extremes, the quotient of this division will be the number sought. As in the thirtyninth question, the order of the terms being 2 acres, 3 horses, 4 days; 5 acres, 6 horses; therefore $\frac{3 \times 4 \times 5}{2 \times 6} = 5$, the number of the days sought.

But here it must be observed, that the terms of this question are capable of being ranged in a different order, thus: If three horses will eat up the grass of two acres of land in four days, how long will six horses be in eating up that of five acres? Here the order of the terms given is, 3 horses, 2 acres, 4 days; 6 horses, 5 acres; according to which order, neither this rule nor the former will solve the question.

Wherefore, to clear this difficulty, and to find out, in what order the Professor's second rule requires the terms given should be placed, it may be useful to investigate a general Theorem for resolving all questions in the Double Rule of Three. This I shall do in the method proposed by the late Mr. WARD of Chester, in his Introduction to the Mathematicks, part 1. chap. 7. sect. 3. pag. 96, where he denotes the six quantities in questions belonging to the Double Rule of Three, by these six letters, P, T, G, p, t, g; the capitals P, T, G denoting the terms in the conditional part of the question, as the small letters p, t, g do the other terms, of the same denomination with their respective capitals. P, p signify the principal terms, or causes of the gain, loss, or expence mentioned in the question; T, t the times (as in quest. 32 and 39) or spaces (as in quest. 33) wherein the said gain, loss, or expence is produced; and G, g the gain, loss, or expence itself, arising from the principals in the aforesaid times or spaces.

Now in order to form a Theorem for resolving questions in the Double Rule of Three direct, let us take any particular question, as for example, the thirty-second: If 2 men in 3 days will earn 4 shillings, how much will 5 men earn in 6 days? Where $2=P$, $3=T$, $4=G$, $5=p$, $6=t$, and g is the number sought. This question then may be resolved into two proportionalities, thus: If 2 (P) men will earn 4 (G) shillings in a given time, 5 (p) men will earn $10 \left(\frac{Gp}{P} \right)$ shillings in the same time. Again, If in 3 (T) days

10 $\left(\frac{Gp}{P}\right)$ shillings be earned, then in 6 (t) days 20 $\left(\frac{Gpt}{PT}\right)$ will be earned.

Therefore $\frac{Gpt}{PT} = g$; and multiplying both sides by PT , we have the Theorem $Gpt = gPT$. And since the thirtysecond question proposed, gives us five terms in this order, P, T, G, p, t , the term sought being $g = \frac{Gpt}{PT}$, it is manifest, that the term sought is found, by dividing the product of the three last terms given, by the product of the two first, according to the Professor's first rule.

Let us now take the thirtyninth question, that we may thence form a Theorem for resolving questions of the same nature with it, which belong to the Double Rule of Three inverse. If 2 (G) acres of land will maintain 3 (P) horses 4 (T) days, how long will 5 (g) acres maintain 6 (p) horses? Which question I resolve into two proportionalities: first into this inverse proportionality, If a certain piece of ground will maintain 3 (P) horses 4 (T) days, it will maintain 6 (p) double the number of horses but half the time, that is, 2 $\left(\frac{PT}{p}\right)$ days: Secondly, into this direct one, If 2 (G) acres will

maintain this double number of horses 2 $\left(\frac{PT}{p}\right)$ days, 5 (g) acres will maintain them 5 $\left(\frac{gPT}{Gp}\right)$ days. Therefore t , the number sought, is $\frac{gPT}{Gp}$.

And multiplying both sides by Gp , we have the same Theorem as before, $Gpt = gPT$, which therefore is a general Theorem for solving all questions in the Double Rule of Three, whether direct or inverse. And since the order of the terms given in the thirtyninth question is G, P, T, g, p ; and since also t , the number sought, is found equal to $\frac{gPT}{Gp}$, therefore according to the Professor's second rule, if the product of the three middle terms be divided by the product of the two extremes, the quotient will be the term sought.

But in the practising of this second rule, care must be always taken to make the term G the first in order: and then, if t be sought, as in the thirtyninth question, the order of the terms given will be G, P, T, g, p , as before; but if p be sought, the order will be G, T, P, g, t . And lastly, I think I scarce need to observe, that if g be the term sought, the operation must always be made by the Professor's first rule, where the order of the terms is P, T, G, p, t , as in the thirtysecond and thirtythird questions; or else T, P, G, t, p , which comes to the same thing.

The

The SUBSCRIBERS omitted in the former List,
are as follows.

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ERRATA.

In the last Page of the former Advertifement, lin. 34, 35. read *compleret*.
* Pag. 120. Prob. 23. lin. 2. for 16 years read 15 years.

POSTU-

P O S T U L A T A.

BEFORE I enter upon my province, it may not be amiss to acquaint my young disciple what preparations he is to make, and what qualifications I expect of him beforehand, that we may neither of us find ourselves disappointed afterwards. I expect then that he knows how to add, to subtract, to multiply, to divide, to find a fourth proportional, and to extract roots, especially the square root: nay I expect further, that he shall not only be able to perform all these operations exactly and readily, but also that he shall be able to apply them upon all common occasions; in a word, I expect that he be tolerably well skilled in common Arithmetick, at least so far as relates to whole numbers: for this reason it is that I have prefixed a few arithmetical questions, wherein he may first try his strength and skill before he ventures any further; they are for the most part very easy, I cannot say indeed they are the best chosen, but they were such as lay in my way when I first begun this work and was hastening to matters of greater moment, and I do not see but they may, if studied with care and attention, answer well enough the end they were intended for: If he finds no difficulty in these, he will have little reason to doubt of his success afterwards; but if he does, he ought then at last to become sensible of his own defects and to endeavour to supply whatever is wanting, and to correct whatever is amiss before he enters himself under my conduct; in the mean time he has my leave to hope that I shall be leis upon the reserve with him when he falls more immediately under my care.

N. B. The *praxis* of the rule of proportion, and of the rule for extracting the square root, not being (properly speaking) of the nature of simple *postulata*, but rather deducible from the four first; I shall not fail to demonstrate these rules so soon as I shall find proper opportunities for that purpose.

Questions for exercise in Multiplication.

Multiplication is taking any one number called the multiplicand as often as is expressed by any other number called the multiplier, and the number produced by this operation is called the product: whence it follows, that the product contains the multiplicand as often as there are units in the multiplier, and that if a number of a greater denomination is to be reduced to an equivalent number of a less, it must be done by multiplication. As for example; In a pound sterling there are 20 shillings; therefore in every sum of money consisting of even pounds, there are twenty times as many shillings as there are pounds; therefore if any number of pounds be multiplied by 20, the product will be an equivalent number of shillings; and the same must be observed in all other cases.

QUEST. 1.

It is required to reduce 456 pounds, 13 shillings and 4 pence into shillings, pence and farthings.

<i>Answer.</i>	Shillings	9133
	Pence	109600
	Farthings	438400.

QUEST. 2.

A certain island contains 36 counties, every county 37 parishes, every parish 38 families, and every family 39 persons: I demand the number of parishes, families and persons in the whole island.

<i>Answer.</i>	Parishes	1332
	Families	50616
	Persons	1974024.

QUEST. 3.

In 1730 years, 42 weeks and 3 days, how many minutes?

N. B. A year consists of 365 days 6 hours, and an hour of 60 minutes.

Hours in one year	8766
In 1730 years	15165180
In 42 weeks 3 days	7128
In the whole	15172308
Minutes in the whole	910338480.

QUEST.

MULTIPLICATION.

3

QUEST. 4.

There is a certain field 102004 feet long, and 102003 feet broad: I demand the number of square feet therein contained.

Answer. 10404714012.

QUEST. 5.

There is a certain floor 24 feet, 4 inches broad, and 96 feet, 6 inches long: I demand how many square inches are therein contained.

Answer. 338136 square inches.

QUEST. 6.

A certain piece of wood 1 foot, 2 inches thick, 3 feet, 4 inches broad, and 5 feet, 6 inches long, is to be cut into small cubes like dies, each of which is to be a quarter of an inch every way: I demand into how many dies the whole may be resolved.

Answer. The whole may be resolved into 2365440 dies.

QUEST. 7.

I demand the number of changes that may be rung on 12 bells.

Changes upon	2 bells	2
	on 3 bells	6
	on 4 bells	24
	on 5 bells	120
	on 6 bells	720
	on 7 bells	5040
	on 8 bells	40320
	on 9 bells	362880
	on 10 bells	3628800
	on 11 bells	39916800
	on 12 bells	479001600.

A 2

QUEST.

QUESTIONS IN

QUEST. 8.

How many different ways can four common dies come up at one throw?

Answer 1296 ways.

QUEST. 9.

Suppose one undertakes to throw an ace at one throw with four common dies; what probability is there of his effecting it?

Answer. By the last question four dies can come up 1296 different ways with and without the ace; and by a like computation, they can come up 625 ways without the ace; therefore there are 671 ways wherein one or more of them may turn up an ace; therefore the undertaker has the better of the lay in the proportion of 671 to 625.

QUEST. 10.

There are two inclosures of the same circumference, that is, both inclosed with the same number of pales; but one is a square whose side is 125 feet, and the other an oblong or long square, 124 feet in breadth, and 126 in length: quære which is the greater close, that is, which, cæteris paribus, will bear most grass.

Answer. The square: for that contains 15625 feet; whereas the other contains but 15624.

Questions for exercise in Division.

The design of division is to shew how often one number called the divisor is contained in another called the dividend, and the number that shews this is called the quotient: whence, and from the definition of multiplication already given, I observe 1st, That the divisor multiplied by the quotient, and consequently the quotient multiplied by the divisor, will always be equal to the dividend, provided there be no remainder after the division is over; but if there be, then this remainder added to, or taken into the product will give the dividend, which is the best proof of division. 2^{dly}, That as the divisor is such a part of the dividend as is expressed by the quotient; so also is the quotient such a part as is expressed by the divisor. Thus 12 divided by 3 quotes 4; therefore 3 is a fourth part, and

DIVISION.

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4 a third part of 12. 3dly, Hence may a number be found that shall be divisible by any two given numbers whatever without remainders, to wit, by multiplying the two given numbers together. Thus if I would have a number that can be divided by both 6 and 9 without any remainders, I multiply 9 by 6, and the product 54 will answer both conditions; though 18 be the least number of that kind. 4thly, Multiplication and division by the same number are the reverse of each other, and so must necessarily have contrary effects: for whereas multiplication increases a number by taking it as often as is expressed by the multiplicator, division (on the contrary) lessens it, by taking only such a part of it as is expressed by the divisor. 5thly, Hence if a number of a lesser denomination be to be changed into an equivalent number of a greater, as farthings into pence, pence into shillings &c, it must be done by division, as the reverse is done by multiplication. 6thly, Whenever it is proposed to know how often one quantity of any kind is contained in another of the same kind, the numbers representing these quantities must be reduced to the same denomination before any division can take place. Thus if I would know how many thirteenpencehalfpennies there are in 20 shillings, I must not only reduce the thirteenpencehalfpenny to 27 halfpence, but also the 20 shillings into 480 halfpence; and then must enquire by division how often 27 halfpence are contained in 480 halfpence, that is, how often 27 is contained in 480; the quotient is 17, and the remainder 21, that is, 21 halfpence; for in all division, the remainder must be of the same denomination with the dividend whereof it is a part; therefore in 20 shillings there are 17 thirteenpencehalfpennies, and 10 pence halfpenny over.

QUEST. II.

It is required to reduce 987654321 farthings into pounds, shillings and pence.

Answer. 987654321 farthings are equivalent to 246913580 pence and 1 farthing; or to 20576131 shillings, 8d. 19; or to 1028806 pounds, 11s. 8d. 19.

QUEST. 12.

One lends me 1296 guineas when they were valued at 1l. 1s. and sixpence apiece: how many must I pay him when they are valued at 1. 1s. apiece?

Answer. 1326 guineas, 18 shillings.

QUEST.

QUESTIONS IN

QUEST. 13.

A certain floor 24 feet 4 inches broad, 96 feet 6 inches long, is to be laid at the rate of 12 pence the square foot: I demand what the whole charge will amount to.

Answer. The floor contains 338136 square inches, or 2348 square feet and 24 square inches; therefore the whole charge amounts to 117 pounds, 8 shillings and two pence.

QUEST. 14.

There is a certain cooler 36 inches deep, 42 inches wide, and 72 inches long: I demand its solid content in English gallons.

Note. An ale gallon is 282 cubic inches.

Answer. The vessel contains 108864 cubic inches, that is, 386 gallons and 12 cubic inches over.

QUEST. 15.

A cubic foot of water weighs 76 pounds, Troy or Roman weight; and air is 860 times lighter than water: I demand the weight of a cubic foot of air.

N. B. A pound Troy contains 12 ounces, one ounce 20 pennyweights, and one pennyweight 24 grains.

Answer. A cubic foot of air weighs Troy weight 1 oz. 1 pwt. 5 gr.

QUEST. 16.

The mean time of a lunation, that is, from new moon to new moon, is 29 days, 12 hours, 44 minutes and 3 seconds; and a Julian year consists of 365 days, 6 hours: I demand then how many lunations are contained in 19 Julian years.

Hours in a Lunation	708
Minutes	42524
Seconds	2551443
Hours in 19 Julian years	166554
Minutes	9993240
Seconds	599594400
Lunations 235; and 1 hour, 28', 15" over.	

QUEST.

THE RULE OF THREE.

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QUEST. 17.

In what time may all the changes on 12 bells be rung, allowing 3 seconds to every round? See Quest. the 7th.

The number of changes on 12 bells 479001600
 The time 1437004800 seconds,
 or 23950080 minutes,
 or 399168 hours,
 or 45 years, 27 weeks, 6 days, 18 hours.

QUEST. 18.

A General of an army distributes 15 pounds, 19 shillings and 2 pence half-penny, among 4 captains, 5 lieutenants and 60 common soldiers, in the manner following: Every captain is to have 3 times as much as a lieutenant, and every lieutenant twice as much as a common soldier: I demand their several shares.

The share of a common soldier 3s. 4d. $\frac{3}{4}$
 of a lieutenant 6s. 9d. $\frac{1}{2}$
 of a captain 1l. os. 4d. $\frac{1}{2}$

*Questions for exercise in the Rule of Three.**And first in the Rule of Three Direct.*

The rule of proportion, or rule of three, or by some the golden rule, is that which teacheth, having three numbers given to find a fourth proportional, that is, to find a fourth number that shall have the same proportion to some one of the numbers given, as is expressed by the other two; and therefore whenever a question is proposed wherein such a fourth proportional is required, that question is said to belong to the rule of proportion. Now in questions of this nature, especially where the numbers given are not merely abstract numbers, but are applied to particular quantities, three things are usually required, to wit, preparation, disposition, and operation.

First as to the preparation, it must be observed that of the three numbers given in the question, two will always be of the same kind, and must be reduced to the same denomination, if they be not so already; and

and if the remaining number be of a mixt denomination, that also must be reduced to some simple one.

Secondly, in disposing the numbers thus prepared, those two that are of the same denomination must be made the first and third numbers in the rule of proportion, and consequently the remaining number must be the second. But here particular care must be taken, that of the two numbers that are of the same denomination, that be made the third in the rule of proportion, upon which the main stress of the question lies, or to which the question more immediately relates, or which contains the demand; and the place of this number being once known, the other two must take their places as above directed. This ordering of the numbers for the operation is commonly called, stating of the question.

Lastly, having thus stated the question, multiply the second and third numbers together; divide the product by the first, and the quotient thence arising will be the fourth number sought; which fourth number, as well as the remainder, if there be any, must always be understood to be of the same denomination with the second. As for example,

QUEST. 19.

A piece of plate weighing 3 pounds, 4 ounces and 5 pennyweights, Troy weight, is valued at 5 shillings and 6 pence an ounce; what is the value of the whole?

Here we have three quantities concerned in the question, viz. 3 pounds, 4 ounces and 5 pennyweights; one ounce; and 5 shillings and 6 pence; whereof the two first, which are of the same kind, must be reduced to the same denomination, and the last to a simple one, thus: for one ounce I write 20 pennyweights; for 3 pounds, 4 ounces and 5 pennyweights, 805 pennyweights; and for 5 shillings and 6 pence, 66 pence; and so the numbers are sufficiently prepared. In the next place I enquire which of the two numbers 20 and 805, which are of the same denomination, is that upon which the main stress of the question lies, and I find it to be 805; for the main business of this question is to enquire into the value of 805 pennyweights of plate; the rest being no more than *data* in order to discover this: So I make 805 my third number, 20 which is a number of the same denomination my first, and 66 my second, and state the question thus; *If 20 pennyweights of plate be worth 66 pence, what will 805 pennyweights of plate be worth?* Now to answer this question, I multiply 805 by 66, and the product is 53130; this I divide by my first number 20, and the quotient is 2656, and there remains 10, that is, 10 pence; therefore to render my quotient more compleat, I bring the remaining 10 pence

THE RULE OF THREE.

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pence into 40 farthings, and so divide again by 20, and find the quotient to be 2, that is, 2 farthings, without any remainder; so the value sought is 2656 pence, 2 farthings; that is, 11 pounds, 1 shilling and 4 pence half-penny.

A demonstration of this Praxis.

Case 1st. Now to demonstrate this manner of operation, I shall resume the foregoing question, but at first under a different supposition, as thus; *If one pennyweight of plate cost 66 pence, what will 805 pennyweights cost?* Here nobody doubts but that upon this supposition, 805 pennyweights will cost 805 times 66 pence, or 66 times 805, that is, 53130 pence; therefore in all instances of this kind, that is, where the first number in the rule of proportion is unity, the fourth number must be found by multiplying the second and third numbers together.

Case 2d. Let us now put the question as it was at first stated, to wit, *If 20 pennyweights of plate be worth 66 pence, what will 805 pennyweights be worth?* Now upon this supposition it is easy to see, that neither 1 pennyweight, nor consequently 805 pennyweights will be worth above a 20th part of what they were in the former case; and therefore we must not now say that 805 pennyweights are worth 53130 pence, but a 20th part of that sum, viz. 2656 pence 2 farthings: and as this way of reasoning will be the same in all other instances, it follows now, that *In the rule of proportion, let the numbers given be what they will, the fourth number must be had by multiplying the second and third numbers together, and dividing the product by the first.* Q. E. D.

QUEST. 20.

How far will one be able to travel in 7 days 8 hours, at the rate of 13 miles every 4 hours, allowing 12 hours to a travelling day?

Answer. 299 miles.

QUEST. 21.

What will 1296 yards of walling amount to, at the rate of 4 shillings and 5 pence a rod, a rod being 5 yards and a half?

Answer. 52 pounds, 8 pence, 3 farthings.

B

QUEST.

QUESTIONS IN

QUEST. 22.

In the mint of England a pound of gold, that is, 11 ounces fine and 1 allay, is at this time coined into 44 guineas and an half: I demand how much sterling a pound of pure gold is worth, observing that the allay is valued at nothing.

Answer. 50 pounds, 19 shillings and 5 pence $\frac{1}{2}$ penny

QUEST. 23.

What is the annual interest of 987 pounds, 6 shillings and 5 pence at the rate of 6 per cent?

Answer. 59 pounds, 4 shillings and 9 pence $\frac{1}{2}$ penny.

QUEST. 24.

The circumference of the earth according to the French mensuration is 123249600 French feet: I demand the same in English miles.

N. B. A thousand French feet are equivalent to 1068 English feet; 3 feet make a yard, and 1760 yards make a mile.

Answer. 131630573 English feet,
or 43876857 yards and 2 feet,
or 24930 miles, 57 yards and 2 feet.

QUEST. 25.

Supposing all things as in the foregoing question, I demand how long a sound will be in passing from pole to pole, upon a supposition that a sound passes over 1142 feet in a second of time.

Answer. 16 hours and 32 seconds.

QUEST. 26.

Monsieur Huygens found that at Paris, the length of a pendulum that swung seconds, was three feet, 8 lines and $\frac{1}{2}$: I demand it's length in English measure.

Note.

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Note. A line is $\frac{7}{8}$ part of an inch, and 1000 French half lines are equivalent to 1068 English $\frac{1}{2}$ lines, as in the 24th question.

Answer. The length in English measure of a pendulum that swings seconds, is 941 English $\frac{1}{2}$ lines; or 39 inches, 2 lines and $\frac{1}{2}$.

QUEST. 27.

I demand in how long a time, a pipe that discharges 15 pints in 2 minutes, 34 seconds, will fill a cistern that is 36 inches deep, 42 inches wide, and 72 inches long. (see question the 14th.)

Answer. In 31707 seconds; or 8 hours, 48'. 27".

For as eight pints make a gallon, so also eight cubic half inches, that is, eight small cubes of half an inch every way make one cubic inch; therefore a pint contains 282 cubic half inches, and fifteen pints 4230; but the whole vessel contains 108864 cubic inches by quest. 14; which are equivalent to 870912 cubic half inches; therefore this question ought to be stated thus;

If 4230 cubic half inches be discharged in 154 seconds of time, in what time will 870912 cubic half inches be discharged? And the answer is,

In 8 hours, 48'. 27", as above.

QUEST. 28.

If a wall 6 feet thick, 9 feet high and 432 feet long, cost 720 pounds in building, what will be the price of a wall of the same materials, that is 12 feet thick, 18 feet high and 576 feet long?

In the former wall are contained 23328 cubic feet; in the latter 124416; therefore the answer to this question is 3840 pounds.

QUEST. 29.

A certain steeple projected upon level ground a shadow to the distance of 57 yards, when a four-foot staff perpendicularly erected cast a shadow of 5 feet 6 inches: what was the height of the steeple?

Answer. 41 yards, 1 foot, 4 inches.

B 2

QUEST.

QUEST. 30.

Two persons A and B make a joint stock; A puts in 372 pounds, and B 496 pounds, for the same time; and they gain 114 pounds, 2 shillings: I demand each mans share of the gain.

Both their stocks make 868 pounds: say then, if 868 pounds stock, bring in 114 pounds, 2 shillings gain, what will 372 pounds, *A*'s part of the stock bring in? *Answer.* 48 pounds, 18 shillings for *A*'s share of the gain; and this subtracted from the whole gain, leaves 65 pounds, 4 shillings, for *B*'s share of the gain.

Note. If there be ever so many partners, their shares of the gain must all be found by the rule of proportion, except the last, which may be had by subtraction; but it would be better to find them all by the rule of proportion, because then, if all the shares when added together, make up the whole gain, it will be an argument that the work is rightly performed.

QUEST. 31.

Two persons A and B make a joint stock; A puts in 496 pounds for 2 months, and B 620 pounds for 3 months; and they gain 456 pounds. What will be each mans share of the gain?

In order to give an answer to this question, it must be considered, that it is the same in the case of trade, as it is in that of money let out to interest, where time is as good as money, that is, whoever lets out 496 pounds for 2 months, is intitled to the same share of the whole gain, as if he had let out twice as much, that is, 992 pounds, for one month: in like manner, he that lets out 620 pounds for 3 months, has a right to the same share of the gain, as if he had let out three times as much, that is, 1860 pounds, for 1 month: substitute therefore these suppositions instead of those in the question, which may safely be done without affecting the conclusion, and then this question will be reduced to the form of the last, without any consideration of the particular quantity of time, thus; *Two merchants A and B make a joint stock; A puts in 992 pounds, and B 1860 pounds for the same time; and they gain 456 pounds. What will be their respective shares of the gain?*

Answer. *A*'s share will be 158 pounds, 12 shillings and 2 pence; and *B*'s, 297 pounds, 7 shillings and 10 pence.

QUEST.

QUEST. 32.

If two men in three days will earn 4 shillings, how much will 5 men earn in 6 days?

This and the following question belong to that which they call the double rule of three, wherein 5 numbers are concerned: these numbers must always be placed as they are in this example, that is, the two last numbers must always be of the same denomination with the two first respectively, and the number sought of the same denomination with the middle one; then may the question be reduced to the single rule of three two ways, either by expunging the first and fourth numbers, or the second and fifth: if you would have the first and fourth numbers expunged, you must argue thus; two men will earn as much in three days, as one man in two times 3, or 6 days; also 5 men will earn as much in 6 days, as one man in 30 days; substitute therefore this supposition and this demand, instead of those in the question, and it will stand thus; If one man in 6 days will earn four shillings, how much will one man earn in 30 days? Which is as much as to say, *If in 6 days a man will earn 4 shillings, how much will he earn in 30 days?*

Answer. 20 shillings.

If you would have the second and fifth numbers expunged, you must argue thus; two men will earn as much in three days, as 3 times two or 6 men in one day; also 5 men will earn as much in 6 days, as 30 men in one day; put then the question this way, and it will stand thus; If 6 men in one day will earn 4 shillings, how much will 30 men earn in one day? That is, *If in any quantity of time 6 men will earn 4 shillings, how much will 30 men earn in the same time?*

Answer. 20 shillings, as before.

Whosoever attends to both these methods of extermination, will easily fall into a third, which includes both the other, and in practice is much better than either of them; for at the conclusion of both operations, the number sought was found by multiplying 30 by 4, and dividing the product by 6: Now if he looks back, and traces out these numbers, he will find that the number 30 came from the multiplication of the two last numbers 5 and 6 together, that 4 was the middle number in the question, and that the divisor 6 was the product of the two first numbers

2 and

2 and 3 multiplied together; therefore *In all questions of this nature, if the three last numbers be multiplied together, and the product be divided by the product of the two first, the quotient will give the number sought, without any further trouble.*

QUEST. 33.

If for the carriage of three hundred weight 40 miles, I must pay 7 shillings and 6 pence, what must I pay for the carriage of 5 hundred weight 60 miles?

Answer. 225 pence, or 18 shillings and 9 pence.

Questions in the rule of three Inverse.

Hitherto we have instanced in the rule of three direct; but there is also another rule of proportion, called the rule of three inverse; which as to the preparation, and disposition of it's numbers, differs nothing from the rule of three direct, but only in the operation; for whereas there, the fourth number was found, by multiplying the second and third numbers together, and dividing by the first; here it is found by multiplying the first and second numbers together, and dividing by the third. All that remains then, is to be able to distinguish, when a question belongs to one rule, and when to the other; in order to which, observe the following directions: *If more requires more, or less requires less, work by the rule of three direct; but if more requires less, or less requires more, work by the rule of three inverse.* The meaning whereof is, that if, when the third number is greater than the first, the fourth must be proportionably greater than the second; or if, when the third number is less than the first, the fourth must be proportionably less than the second, the question then belongs to the rule of three direct: But if, when the third number is greater than the first, the fourth must be less than the second; or when the third number is less than the first, the fourth must be greater than the second; in either of these cases, the question belongs to the rule of three inverse, and must be resolved as above directed.

As for example,

QUEST. 34.

If 12 men will eat up a quantity of provision in 15 days, how long will 20 men be in eating up the same?

This

THE RULE OF THREE.

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This question is of such a nature, that more requires less; for 20 men will consume the same provision in less time than 12; therefore the question belongs to the rule of three inverse; so I multiply the first and second numbers together, and divide by the third, and the quotient 9, that is, 9 days, is an answer to the question.

A demonstration of the rule of three Inverse.

If I was to answer this question by pure dint of thought, without any rule to direct me, I should reason thus: whatever quantity of provision lasts 12 men 15 days, the same will last 1 man 12 times as long, that is, 12 times 15, or 180 days; but if it will last 1 man 180 days, it will last 20 men but the 20th part of that time, that is, 9 days: here then the fourth number was found by multiplying the first and second numbers together, and dividing the product by the third; and the reason is the same in all other cases, wherever the rule of three inverse is concerned.
Q. E. D.

QUEST. 35.

One lends me 372 pounds for 7 years and 8 months, or 92 months: how long must I lend him 496 pounds for an equivalent?

Answer. 5 years, 9 months.

QUEST. 36.

If a square pipe 4 inches and 5 lines wide, will discharge a certain quantity of water in one hour's time; in what time will another square pipe, 1 inch and 2 lines wide, discharge the same quantity from the same current?

The orifice of a square pipe 4 inches, 5 lines, or 53 lines wide, contains 2809 square lines; and the orifice of a pipe 1 inch, 2 lines, or 14 lines wide, contains 196 square lines. Say then, *If an orifice of 2809 square lines will discharge a certain quantity of water in one hour; in what time will an orifice of 196 square lines discharge the same?*

Answer. In 14 hours, 19'. 54".

QUEST.

QUEST. 37.

If 3 men, or 4 women, will do a piece of work in 56 days, how long will one man and one woman be in doing the same?

Because of the 3 men, or 4 women, some number must be found that is divisible both by 3 and by 4 without remainder; such a one is the number 12, which is the product of 3 and 4 multiplied together; (see observation the third upon the definition of division :) make then 3 men or 4 women equivalent to 12 boys, and you will have 1 man equivalent to 4 boys, 1 woman to 3 boys, and 1 man and 1 woman to 7 boys, and the question will stand thus; *If 12 boys will do a piece of work in 56 days, how long will 7 boys be in doing the same?*

Answer. 96 days.

QUEST. 38.

If 5 oxen, or 7 colts, will eat up a close in 87 days, in what time will 2 oxen and 3 colts eat up the same?

Answer. In 105 days.

QUEST. 39.

If 2 acres of land will maintain 3 horses 4 days, how long will 5 acres maintain 6 horses?

This question may perhaps at first sight, be taken to be somewhat of the same nature with the 32d and 33d questions, which belonged to the double rule of three direct; but when it comes to be examined into more narrowly, it will be found to be of a very different nature: for we cannot say here as we did there, that 2 acres will last 3 horses as long as 1 acre will last 6 horses; this would be a very unjust way of thinking, and wherever it is so, the question ought to be referred to another rule, which they call the double rule of three inverse; the propriety or impropriety of this thought, being an infallible criterion whereby to distinguish, when a question belongs to one rule, and when to the other. All questions belonging to this rule, as well as those belonging to the other, may be reduced to the single rule of three two ways; either by expunging the first and fourth numbers, or the second and fifth; but then the methods
of

THE RULE OF THREE.

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of extermination are different. In questions of this nature, if the first and fourth numbers are to be expunged, the 2 first numbers are to be multiplied by the fourth, and the 2 last by the first; but if the second and fifth numbers are to be expunged, then the two first numbers are to be multiplied by the fifth, and the two last by the second: thus in the question before us, if we would exterminate the first and fourth numbers, we must multiply the two first numbers, that is, 2 and 3, by the fourth, that is, by 5, and say, that 2 acres will last 3 horses just as long as 10 acres will last 15 horses; we must also multiply the 2 last numbers, to wit, 5 and 6, by the first, that is, by 2, and say, that 5 acres will last 6 horses as long as 10 acres will last 12 horses: use now these numbers instead of those in the question, and it will be changed into this equivalent one; If 10 acres of land will maintain 15 horses 4 days, how long will 10 acres maintain 12 horses? Strike out of the question the first and fourth numbers, which being equal, will be of no use in the conclusion, and then the question will stand thus; *If 15 horses will eat up a certain piece of ground in 4 days, how long will 12 horses be in eating up the same?*

Answer. 5 days; for this question belongs to the rule of three inverse.

If we would exterminate the second and fifth numbers out of the question, we must multiply the two first numbers by the fifth, and say, that 2 acres will last 3 horses just as long as 12 acres will last 18 horses; we must also multiply the 2 last numbers by the second, and say, that 5 acres will last 6 horses as long as 15 acres will last 18 horses: use these numbers instead of those in the question, and it will be changed into this equivalent one; If 12 acres will maintain 18 horses 4 days, how long will 15 acres maintain 18 horses? That is, (striking out the second and fifth numbers) *If 12 acres of land will maintain a certain number of horses 4 days, how long will 15 acres last the same number?*

Answer. 5 days, as before; for this question belongs to the rule of three direct.

In both these operations, the number sought was at last found by multiplying 15 by 4, and then dividing the product by 12: now whosoever looks back upon the foregoing resolution, and observes how these numbers were formed, he will easily perceive, that the number 4 was the middle term in the question; that the number 15 in both operations was the product of the numbers 3 and 5, which lay next the middle term on each side; and that the divisor 12 was in both cases the product of the extreme numbers 2 and 6: therefore *In all questions be-*

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longing

longing to the double rule of three inverse, where the numbers are supposed to be ordered as in the double rule of three direct, if the three middle numbers be multiplied together, and the product be divided by the product of the two extremes, the quotient of this division will be the number sought. And thus may all the trouble of expunging be avoided, though I thought it proper to explain that method in the first place, in order to let the learner into the reason of this last theorem which is founded upon it.

Questions wherein the extraction of the square root is concerned.

QUEST. 40.

There is a certain field whose breadth is 576 yards, and whose length is 1296 yards: I demand the side of a square field equal to it.

Answer. This field will be equal to a square whose side is 864 yards.

QUEST. 41.

There is a certain inclosure 3 times as long as it is broad, whose area is 46128 square yards: I demand its breadth and length.

The breadth multiplied into the length, that is, the breadth multiplied into 3 times itself, is 46128; therefore the breadth multiplied into itself is 15376; therefore the breadth is 124, and the length 372.

QUEST. 42.

A certain society collect among themselves a sum amounting to 15 pounds, 5 shillings and a farthing, every one contributing as many farthings as there were members in the whole society: I demand the number of members.

Answer. 121 members.

THE

THE INTRODUCTION,

Concerning Vulgar and Decimal Fractions.

DEFINITIONS.

Art. 1. **A** FRACTION simply and abstractedly considered, is that wherein some part or parts of an unit are expressed: as if an unit be supposed to be divided into 4 equal parts, and three of these parts are to be expressed, it must be done by the fraction three fourths, to be written thus $\frac{3}{4}$: here the number 4, which shews into how many equal parts the unit is supposed to be divided, and so determines the true value, magnitude, or denomination of those parts, is called the denominator of the fraction; and the number 3, which shews how many of these parts are considered in the fraction, is called the numerator: thus in the fraction $\frac{1}{2}$ or one half, 1 is the numerator, and 2 the denominator; in $\frac{2}{2}$ or two halves, 2 is both numerator and denominator, &c.

When a fraction is applied to any particular quantity, that quantity is called the integer to the fraction: thus in $\frac{3}{4}$ of a penny, a penny is the integer; in three fourths of six, the number 6 is the integer; thus in three fourths of five sixths, the fraction five sixths is the integer; for though in an absolute sense it be a fraction, yet here with respect to the fraction three fourths, it is an integer: and thus may one and the same quantity, under different ways of conception, be both an integer and a fraction; as a foot is an integer, and a third part of a yard is a fraction, though they both signify the same thing. When the integer to a fraction is not expressed, unity is always to be understood: thus $\frac{3}{4}$ is $\frac{3}{4}$ of an unit; thus when we say, $\frac{1}{3}$ and $\frac{1}{4}$ make $\frac{7}{12}$, the meaning is, that if $\frac{1}{3}$ part of an unit, and $\frac{1}{4}$ part of an unit be added together, the sum will amount to the same as if that unit had been divided into 12 equal parts, and 7 of those parts had been taken; thus again, when we say that $\frac{2}{3}$ of $\frac{4}{5}$ are equivalent to $\frac{8}{15}$, we mean, that if an unit be divided into 5 equal parts, and 4 of them be taken, and then this fraction $\frac{4}{5}$ be again divided into 3 equal parts,

parts, and 2 of them be taken, the result will be the same, as if the unit had at first been divided into 15 equal parts, and 8 of them had been taken; and whatever is true in the case of unity, will be equally true in the case of any other integer whatever: thus if it be true that $\frac{1}{3}$ and $\frac{1}{4}$ of an unit are equal to $\frac{7}{12}$ of an unit, that is, if it be true in general that $\frac{1}{3}$ and $\frac{1}{4}$ added together are equal to $\frac{7}{12}$, it will be as true of any particular integer, suppose of a pound sterling, that $\frac{1}{3}$ of a pound, and $\frac{1}{4}$ of a pound when added together, are equal to $\frac{7}{12}$ of a pound; again, if it be true in general that $\frac{2}{3}$ of $\frac{4}{5}$ are equal to $\frac{8}{15}$, it is as true in particular that $\frac{2}{3}$ of $\frac{4}{5}$ of a pound are equivalent to $\frac{8}{15}$ of a pound, &c.

Of proper and improper fractions; and of the reduction of an improper fraction to a whole or mixt number.

2. Fractions are of two sorts, proper and improper; a proper fraction is that, whose numerator is less than the denominator, as $\frac{1}{2}$; therefore an improper one is that, whose numerator is equal to, or greater than the denominator, as $\frac{2}{2}$, $\frac{3}{2}$ &c.

O B J E C T I O N.

But is there no absurdity in the supposition of an improper fraction, as in three halves for instance, considering that an unit cannot be divided into more than two halves? *Answer*: no more than there is in supposing three half pence to be the price of any thing, considering that a penny cannot be divided into above two halfpence. These fractions therefore are called improper, not from any absurdity either in the supposition or in the expression, but because they may be more properly and more intelligibly expressed, either by a whole number, or at least by a mixt number consisting of a whole number and a fraction; as for example, if the numerator of a fraction be equal to the denominator, as $\frac{4}{4}$, that fraction will always be equivalent to unity, as $\frac{4}{4}$ of an hour, that is, four quarters of an hour, are equivalent to one hour, $\frac{4}{4}$ of a penny, that is, four farthings, are equal to one penny, &c: and the reason is plain; for if an unit be divided into four equal parts, and four of these parts be expressed in a fraction, the whole unit is expressed in that fraction, that is, such a fraction must always be looked upon as equal to an unit: therefore if the numerator be double of the denominator, as $\frac{8}{4}$, the fraction must be equal to the number 2, because $\frac{8}{4}$ contain $\frac{4}{4}$ or 1 twice; in like manner $\frac{12}{4}$ are equal to, and may be more properly expressed by, the number 3; $\frac{16}{4}$ by the number 4, &c: and universally, as often as the numerator

Art. 2, 3. REDUCTION OF FRACTIONS. 21

numerator of a fraction contains the denominator, so many units is that fraction equivalent to : but to find how often the numerator contains the denominator, is to divide the numerator by the denominator ; therefore if the numerator of an improper fraction be divided by the denominator, the quotient, if nothing remains, will be the whole number by which the fraction may be expressed ; but if any thing remains of this division, then the quotient together with a fraction whose numerator is that remainder, and denominator the divisor, will be a mixt number, expressing the fraction proposed : thus $\frac{24}{3}$ are equivalent to the whole number 8, but $\frac{25}{3}$ are equivalent to the mixt number $8\frac{1}{3}$, $\frac{26}{3}$ to the mixt number $8\frac{2}{3}$, just as 24 feet are equal to 8 yards, 25 feet to 8 yards and 1 foot, 26 feet to 8 yards and 2 feet, &c : and this is what we call the reduction of an improper fraction into a whole or mixt number.

The reduction of a whole or mixt number into an improper fraction.

3. As unity may be expressed by any fraction of any form or denomination whatever, provided the numerator be equal to the denominator, as $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, &c ; so the number 2 is reducible to any fraction whose numerator is double the denominator, as $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, &c ; and so is every number reducible to any fraction, whose numerator contains the denominator as often as there are units in the number proposed : therefore whenever a whole number is to be reduced to a fraction whose denominator is given, it must be multiplied by that given denominator, and the product with that denominator under it, will be the equivalent fraction ; thus if the number 5 is to be reduced into halves, that is, into a fraction whose denominator is 2, it must be multiplied by 2, and so you will have 5 equal to $\frac{10}{2}$, just as 5 pence are equivalent to 10 halfpence ; if the number 8 is to be reduced into thirds, it must be multiplied by 3, and so you will have 8 equal to $\frac{24}{3}$, just as 8 yards are equal to 24 feet ; lastly, if the number 2 is to be reduced into fourths, it will be equal to $\frac{8}{4}$, just as 2 pence are equal to 8 farthings. If the number to be reduced be a mixt number, consisting of a whole number and a fraction, the whole number must always be reduced to the same denomination with the fraction annexed, and the rule will be this : multiply the whole number by the denominator of the fraction annexed ; add the numerator to the product, and the sum with the denominator under it will be the equivalent fraction : Thus the mixt number $5\frac{1}{2}$ is equivalent to $\frac{11}{2}$, just as 5 pence halfpenny in money is equivalent to 11 halfpence : This operation carries it's own evidence along with it ; for the number 5 itself is equal to $\frac{10}{2}$ as.

as above; therefore $5\frac{1}{2}$ must be equivalent to $\frac{11}{2}$: again, the number $8\frac{1}{3}$ is equal to $\frac{25}{3}$, just as 8 yards and 2 feet over are equivalent to 26 feet; lastly, $2\frac{3}{4}$ is reducible to $\frac{11}{4}$, just as 2 pence and 3 farthings are reducible to 11 farthings.

A L E M M A.

4. *If any integer be assumed, as a pound sterling, and also any fraction, as $\frac{1}{4}$, I say then, that $\frac{1}{4}$ parts of one pound, amount to the same as $\frac{1}{4}$ part of 3 pounds.*

To demonstrate this Lemma (which scarce wants a demonstration) I argue thus: If any quantity, greater or less, be always divided into the same number of parts, the greater or less the quantity so divided is, the greater or less will the parts be; thus $\frac{1}{4}$ of a yard is 3 times as much as $\frac{1}{4}$ of a foot, because a yard is 3 times as much as a foot; and for the same reason $\frac{1}{4}$ of 3 pounds is 3 times as much as $\frac{1}{4}$ of 1 pound; but $\frac{3}{4}$ of 1 pound are also 3 times as much as $\frac{1}{4}$ of 1 pound; therefore $\frac{3}{4}$ of 1 pound are equal to $\frac{1}{4}$ of 3 pounds, because both are just 3 times as much as $\frac{1}{4}$ of 1 pound.
Q. E. D.

How to estimate any fractional parts of an integer in parts of a lesser denomination, and vice versa.

5 This may be done various ways; but the shortest and safest, as I take it, is that which follows: Suppose I had a mind to know the value of $\frac{1}{6}$ of a pound; I should argue as in the foregoing lemma, that $\frac{1}{6}$ of one pound, are the same as $\frac{1}{6}$ of 5 pounds; but the latter is more easily taken than the former; therefore I apply myself wholly to the latter, to wit, to find the sixth part of 5 pounds, thus: 5 pounds, or 100 shillings, divided by 6, quote 16 shillings, and there remain 4 shillings; again, 4 shillings, or 48 pence, divided by 6, quote 8 pence, and there remains nothing; therefore the value of 1 sixth of 5 pounds, or $\frac{1}{6}$ of 1 pound, is 16 shillings and 8 pence. Again, suppose I would know the value of $\frac{1}{7}$ of a pound, I find the value of $\frac{1}{7}$ of 6 pounds thus; 6 pounds, or 120 shillings divided by 7, give 17 shillings, and there remains 1 shilling; again, 1 shilling, or 12 pence, divided by 7, give 1 penny, and there remain 5 pence; again, 5 pence, or 20 farthings, divided by 7, give 2 farthings, and there remain 6 farthings; lastly, a seventh part of 6 farthings is just as much as $\frac{6}{7}$ of 1 farthing, by the lemma: hence I conclude, that $\frac{1}{7}$ of a pound are 17 shillings, 1 penny, 2 farthings, and $\frac{6}{7}$ of a farthing: But the value of $\frac{6}{7}$ of a farthing is so near to one farthing, that if I would rather

rather admit of a small inaccuracy in my account, than a fraction, I should make the value of $\frac{6}{7}$ of a pound to be 17 shillings, 1 penny and 3 farthings. Lastly, suppose I would know the amount of $\frac{2}{3}$ parts of 17 shillings and sixpence, I should argue thus; $\frac{2}{3}$ parts of 17 shillings and sixpence, are equivalent to $\frac{1}{3}$ part of twice as much, that is, to $\frac{1}{3}$ part of 35 shillings: but $\frac{1}{3}$ part of 35 shillings is 11 shillings and 8 pence; therefore $\frac{2}{3}$ parts of 17 shillings and sixpence make 11 shillings and 8 pence.

Of the reverse of this reduction, one single instance will suffice: Let it then be required to reduce 1 shilling, 2 pence, 3 farthings, to fractional parts of a pound: here I consider, that in 1 pound are 960 farthings; and in 1 shilling, 2 pence, 3 farthings, are 59 farthings; therefore 1 farthing is $\frac{1}{960}$ of a pound; and 1 shilling, 2 pence, 3 farthings, are $\frac{59}{960}$ of a pound.

*Preparations for further reductions and operations
of fractions.*

6 All the operations and reductions of fractions, are mediately or immediately deducible from the following principle, which is; that *If the numerator of a fraction be increased, whilst the denominator continues the same, the value of the fraction will be increased proportionably; and vice versâ. On the other hand, if the denominator be increased in any proportion, whilst the numerator continues the same, the value of the fraction will be diminished in a contrary proportion; and vice versâ.* Thus $\frac{2}{3}$ are twice as much as $\frac{1}{3}$, and $\frac{1}{6}$ is but half as much.

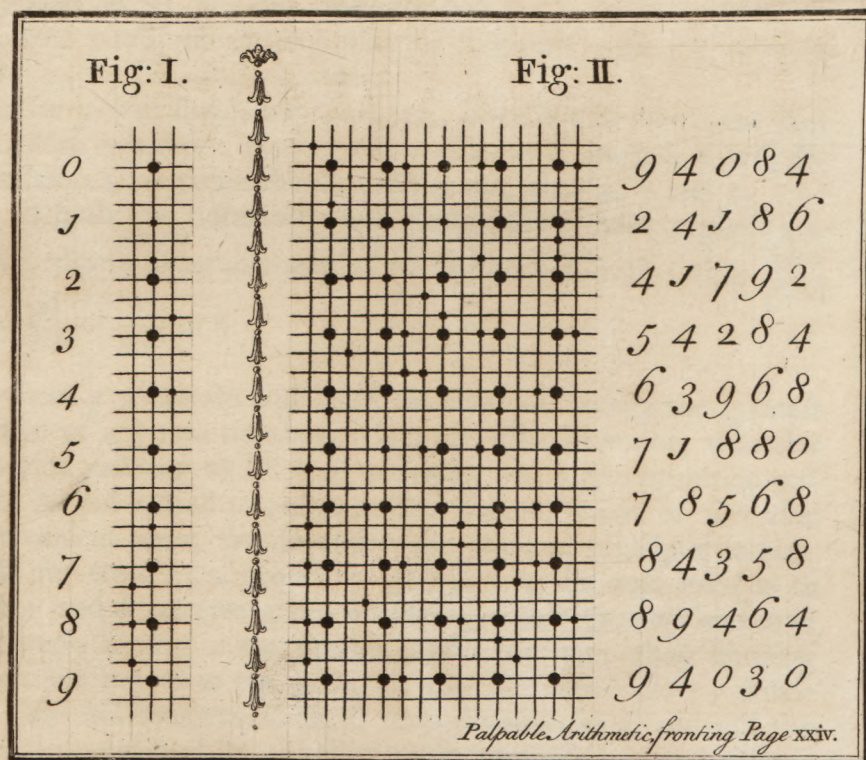
From this principle it follows, that if the numerator and denominator of a fraction be both multiplied, or both divided by the same number, the value of the fraction will not be affected thereby; because, as much as the fraction is increased by multiplying the numerator, just so much again it will be diminished by multiplying the denominator; and as much as the fraction is diminished by dividing the numerator, just so much again it will be increased by dividing the denominator: Thus the terms of the fraction $\frac{1}{4}$ being doubled, produce $\frac{2}{8}$, a fraction of the same value; and on the contrary, the terms of the fraction $\frac{6}{8}$ being halved, give $\frac{3}{4}$.

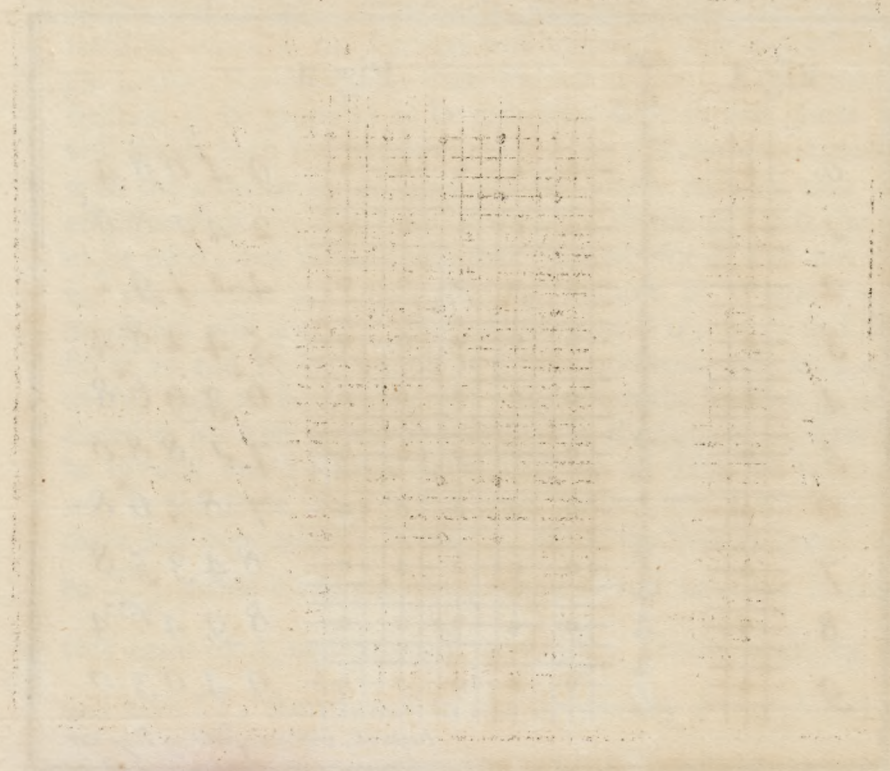
Hence it appears, that every fraction is capable of infinite variety of expression, since there is infinite choice of multipliers, whereby the numerator and denominator of a fraction may be multiplied, and so the expression may be changed, without changing the value of the fraction: thus the fraction $\frac{1}{2}$, if both the numerator and denominator be multiplied by 2, becomes $\frac{2}{4}$; if by 3, $\frac{3}{6}$; if by 4, $\frac{4}{8}$; if by 5, $\frac{5}{10}$; and so on *ad infinitum*; all which are nothing else but different expressions of the same fraction:

fraction: therefore in the midst of so much variety, we must not expect that every fraction we meet with should always be in it's least or lowest terms; but how to reduce them to this state whenever they happen to be otherwise, shall be the business of the next article.

The reduction of fractions from higher to lower terms.

7. Whenever a fraction is suspected not to be in it's least terms, find out, if possible, some number that will divide both the numerator and denominator of the fraction without any remainder; for if such a number can be found, and the division be made, the two quotients thence arising will exhibit respectively, the numerator and denominator of a fraction, equal to the fraction first proposed, but expressed in more simple terms: this is evident from the last article. As for example; let the fraction $\frac{10}{15}$ be proposed to be reduced: here to find some number that will divide both the numbers 10 and 15 without any remainder, I begin with the number 2, as being the first whole number that can have any effect in division; but I find 2 will not divide 15; 3 is the next number to be tried; but neither will that succeed, for it will not divide 10; as for the number 4, I pass that by, because if 2 would not divide 15, much less will 4 do it; the next number I try is 5, and that succeeds; for if 10 and 15 be divided by 5, the quotients will be 2 and 3 respectively, each without remainder; therefore the fraction $\frac{10}{15}$, after being reduced to it's least terms, is found to be the same as $\frac{2}{3}$; that is, if an unit be divided into 15 equal parts, and 10 of them be taken, the amount will be the same, as if it had been divided into 3 equal parts, and 2 of them had been taken. Secondly, if the fraction proposed to be reduced be $\frac{2520}{7560}$, divide it's terms by 2, and you will have the fraction $\frac{1260}{3780}$; divide again by 2, and you will have $\frac{630}{1890}$; divide again by 2, and you will have $\frac{315}{945}$; therefore all further division by 2 is excluded: divide then these last terms by 3, and you will have $\frac{105}{315}$; divide again by 3, and you will have $\frac{35}{105}$; divide by 5, and you will have $\frac{7}{21}$; and lastly divide by 7, and you will have $\frac{1}{3}$; so that the fraction $\frac{2520}{7560}$, after a common division by 2, 2, 2, 3, 3, 5, 7, is found at last equal to $\frac{1}{3}$. Thirdly, the fraction $\frac{36}{48}$, after a continual division by 2, 2, 3, becomes $\frac{3}{4}$. Fourthly, $\frac{56}{24}$ after a continual division by 2, 2, 7, becomes $\frac{2}{3}$. Fifthly, $\frac{144}{180}$ after a conti-





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continual division by 2, 2, 3, 3, becomes $\frac{4}{3}$. Sixthly, $\frac{42}{126}$ after a continual division by 2, 3, and 7, becomes $\frac{1}{3}$. Seventhly, $\frac{315}{840}$ after a continual division by 3, 5, 7, becomes $\frac{1}{8}$. Eighthly, $\frac{35}{840}$ after a continual division by 5 and 7, becomes $\frac{1}{24}$. Ninthly, $\frac{735}{245}$ after a continual division by 5, 7, 7, becomes $\frac{1}{1}$, or 3.

Some perhaps may think themselves helped in the practice of this rule by the following observations:

First, that 2 will divide any number that ends with an even number, or with a cypher, as 36, 30, &c. and no other.

Secondly, that 5 will divide any number that ends with a 5, or with a cypher, as 75, 70, &c. and no other.

Thirdly, that 3 will divide any number, when it will divide the sum of it's digits added together: thus 3 will divide 471, because it will divide the number 12, which is the sum of the numbers 4, 7 and 1.

Fourthly, if both the numerator and denominator have cyphers annexed to them, throw away as many as are common to both: thus $\frac{3500}{56000}$ is the same as $\frac{35}{560}$, or $\frac{7}{112}$, or $\frac{1}{16}$.

After all, there is a certain and infallible rule for finding the greatest common divisor of any two numbers whatever, that have one, whereby a fraction may be reduced to it's least terms by one single operation only. I shall be forced indeed to postpone the demonstration of this rule to a more convenient place, not so much for want of principles to proceed upon, as for want of a proper notation; but the rule itself is as follows: Let a and b be two given numbers, whose greatest common divisor is required; to wit, a the greater, and b the less: then dividing a by b without any regard to the quotient, call the remainder c ; divide again b by c , and call the remainder d ; then divide c by d , and call the remainder e ; then divide d by e , and call the remainder f ; and so proceed on, till at last you come to some divisor, as f , which will divide the preceding number e without a remainder: I say then, that this last divisor will be the greatest common divisor of the two given numbers a and b . As for example; let a be 1344 and b 582: then to find the greatest common divisor of these numbers, I divide a (1344) by b (582) and there remains 180, which I call c ; then I divide b (582) by c (180) and there remains 42, which I call d ; then I divide c (180) by d (42) and there remains 12, which I call e ; then I divide d (42) by e (12) and there remains 6, which I call f ; lastly I divide e (12) by f (6) and

D

and there remains nothing: whence I conclude that 6 is the greatest common divisor of the two numbers 1344 and 582; and as the quotients by 6 are 224 and 97, it follows, that the fraction $\frac{582}{1344}$, when reduced to it's least terms, will be $\frac{97}{224}$. If no common divisor can be found but unity, it is an argument that the fraction is in it's least terms already.

From this and the last article it follows, that all fractions that are reducible to the same least terms are equal; as $\frac{4}{8}$, $\frac{6}{9}$, $\frac{10}{15}$, &c, which are all reducible to $\frac{2}{3}$; though it does not follow *à converso*, that all equal fractions are reducible to the same least terms; this will be demonstrated in another place. (See *Elements of Algebra*, Art. 193.)

For the better understanding of the following article, it must be observed, that this mark \times is a sign of multiplication, and is usually read *into*: thus 2×3 signifies 6, $2 \times 3 \times 4$ signifies 24, $2 \times 3 \times 4 \times 5$ signifies 120, &c; and in some cases it will be better to put down these components or factors, than the character of the number arising from their continual multiplication, as in the following article. It ought also to be observed, that it matters not in what order these components are placed; for $2 \times 3 \times 4 \times 5$ signifies just the same as $4 \times 5 \times 2 \times 3$, &c.

The reduction of fractions of different denominations, to others of the same denomination.

8. There is another reduction of fractions, no less useful than the former; and that is, the reduction of fractions of different denominations to others of the same denomination, or which have the same denominator, without changing their values; which is done as follows: Having first put down the fractions to be reduced, in any order, one after another, and beginning with the numerator of the first fraction, multiply it by a continual multiplication, into all the denominators but it's own, and put down the product under that fraction; then multiply in like manner, the numerator of the next fraction into all the denominators but it's own, and put down the product under that fraction; and so proceed on through all the numerators, always taking care to except the denominator of that fraction, whose numerator is multiplied. Then multiplying all the denominators together, put down the product under every one of the products last found, and you will have a new set of fractions, all of the same denomination with one another, and all of the same values with their respective original ones. As for example; let it be proposed to reduce the following fractions to the same denomination, $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{7}{8}$: *1st*, The numerator of the first fraction is 1, and the denominators

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FRACTIONS.

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nators of the rest are 4, 6 and 8, and $1 \times 4 \times 6 \times 8$ gives 192; therefore I put down 192 under $\frac{1}{2}$. 2dly, The numerator of the second fraction is 3, and the denominators of the rest are 6, 8 and 2, and $3 \times 6 \times 8 \times 2$ gives 288; therefore I put down 288 under $\frac{3}{4}$. 3dly, $5 \times 8 \times 2 \times 4$ gives 320; therefore I put down 320 under $\frac{5}{6}$. 4thly, $7 \times 2 \times 4 \times 6$ gives 336; therefore I put down 336 under $\frac{7}{8}$. Lastly, $2 \times 4 \times 6 \times 8$, or the product of all the denominators is 384: this therefore I put down under every one of the numerators last found, and so have a new set of fractions, viz. $\frac{192}{384}, \frac{288}{384}, \frac{320}{384}, \frac{336}{384}$, all of the same denomination, as appears from the operation itself; and all of the same value with their respective original ones, as will appear presently; but first see the work:

$$\begin{array}{cccc} \frac{1}{2} & \frac{3}{4} & \frac{5}{6} & \frac{7}{8} \\ \frac{192}{384} & \frac{288}{384} & \frac{320}{384} & \frac{336}{384} \end{array}$$

A demonstration of the rule.

All that is to be demonstrated in this rule is, to prove from the nature of the operation itself, that the original fractions suffer nothing in their values by this reduction: in order to which, it will be convenient to put down the components of the new numerators instead of their proper characters, as in the last article; as also those of the common denominator, and the work will stand thus:

$$\begin{array}{cccc} \frac{1}{2} & \frac{3}{4} & \frac{5}{6} & \frac{7}{8} \\ \frac{1 \times 4 \times 6 \times 8}{2 \times 4 \times 6 \times 8} & \frac{3 \times 6 \times 8 \times 2}{4 \times 6 \times 8 \times 2} & \frac{5 \times 8 \times 2 \times 4}{6 \times 8 \times 2 \times 4} & \frac{7 \times 2 \times 4 \times 6}{8 \times 2 \times 4 \times 6} \end{array}$$

By this method of operation it appears, that the numerator and denominator of the first fraction $\frac{1}{2}$, are both multiplied by the same number in the reduction, to wit, by $4 \times 6 \times 8$; and therefore that fraction suffers nothing in it's value, by art. 6. In like manner, the terms of the second fraction $\frac{3}{4}$ are both multiplied by the same number $6 \times 8 \times 2$; therefore that fraction can suffer nothing in it's value; and the same may be said of all the rest. Q. E. D.

Other examples to this rule.

$$\begin{array}{ccccccccc} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{360}{720} & \frac{240}{720} & \frac{180}{720} & \frac{144}{720} & \frac{120}{720} & \frac{120}{360} & \frac{90}{360} & \frac{72}{360} & \frac{60}{360} \end{array}$$

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$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{6}$
$\frac{30}{120}$	$\frac{24}{120}$	$\frac{20}{120}$	$\frac{6}{30}$	$\frac{5}{30}$

The use of this rule will soon appear in the addition and subtraction of fractions: in the mean time it may not be amiss to observe, that it would be very difficult, if not impossible, to compare fractions of different denominations, without first reducing them to the same. As for instance; suppose it should be asked, which of these two fractions is the greater, $\frac{3}{4}$, or $\frac{5}{7}$; in this view it would be difficult to determine the question; but when I know that $\frac{3}{4}$ are the same with $\frac{21}{28}$, and that $\frac{5}{7}$ are the same with $\frac{20}{28}$, I know then, that $\frac{3}{4}$ are greater than $\frac{5}{7}$ by a twenty eighth part of the whole. We now proceed to the four operations of fractions, to wit, their addition, subtraction, multiplication and division: and first,

Of the addition of fractions.

9. Whenever two or more fractions are to be added together, let them first be reduced to the same denomination, if they be not so already; and then adding the new numerators together, put down the sum with the common denominator under it. In the case of mixt numbers, add first the fractions together, and then the whole numbers: but if the fractions when added together, make an improper fraction, reduce it by the 2d art. to a whole, or mixt number; and then putting down the fractional part, if there be any, reserve the whole number for the place of integers.

To this rule might be referred (if it had not been taught already in the 3d art.) the reduction of a mixt number into an improper fraction, which is nothing else but adding a whole number and a fraction together, and may be done by considering the whole number as a fraction whose denominator is unity.

Examples of addition in fractions.

1st, $\frac{3}{10}$ and $\frac{4}{10}$ when added together make $\frac{7}{10}$, for just the same reason as 3 shillings and 4 shillings when added together make 7 shillings.

2^{dly}, The fractions $\frac{1}{3}$ and $\frac{1}{4}$ when reduced to the same denomination by the last art. are $\frac{4}{12}$ and $\frac{3}{12}$, and these added together make $\frac{7}{12}$; therefore the fractions $\frac{1}{3}$ and $\frac{1}{4}$ when added together make up the fraction $\frac{7}{12}$.

For a better confirmation of these abstract conclusions, but chiefly to inure the learner to conceive and reason distinctly about fractions, it will be

be very convenient to apply these examples in some particular case, as for instance, in the case of a pound sterling; and if we do so here, we are to try, whether $\frac{1}{3}$ and $\frac{1}{4}$ of a pound when added together, amount to $\frac{7}{12}$ of a pound, or not: here then we shall find by division, that the third part of a pound is 6 shillings and 8 pence, and the fourth part 5 shillings; and these added together, make 11 shillings and 8 pence; therefore $\frac{1}{3}$ and $\frac{1}{4}$ of a pound when added together, make 11 shillings and 8 pence; but by the 5th art. it will be found that $\frac{7}{12}$ of a pound are also 11 shillings and 8 pence; therefore $\frac{1}{3}$ and $\frac{1}{4}$ of a pound, when added together, make $\frac{7}{12}$ of a pound; and the same would have been true in any other instance whatever.

3dly, $\frac{2}{5}$ and $\frac{3}{8}$, that is, $\frac{16}{40}$ and $\frac{15}{40}$, when added together, make $\frac{31}{40}$, which will also be true in the case of a pound sterling; for by the 5th art. $\frac{2}{5}$ of a pound are 8 shillings, $\frac{3}{8}$ of a pound are 7 shillings and 6 pence, and their sum is 15 shillings and 6 pence; which will also be found to be the value of $\frac{31}{40}$ of a pound; therefore $\frac{2}{5}$ and $\frac{3}{8}$ of a pound when added together, make $\frac{31}{40}$ of a pound.

4thly, $\frac{2}{3}$ and $\frac{4}{5}$, that is, $\frac{10}{15}$ and $\frac{12}{15}$, when added together, make $\frac{22}{15}$, an improper fraction; which being reduced to a mixt number, by the 2d art. is 1 and $\frac{7}{15}$: let us now try, whether $\frac{2}{3}$ of a pound, and $\frac{4}{5}$ of a pound when added together will make one pound and $\frac{7}{15}$ of a pound over, or not: now $\frac{2}{3}$ of a pound, or 13 shillings and 4 pence, added to $\frac{4}{5}$ of a pound, or 16 shillings, amount to 1 pound 9 shillings and 4 pence; and $\frac{7}{15}$ of a pound, are found to be 9 shillings and 4 pence; therefore $\frac{2}{3}$ and $\frac{4}{5}$ of a pound when added together, make one pound and $\frac{7}{15}$ of a pound over.

5thly, $\frac{3}{4}$ and $\frac{5}{6}$, that is, $\frac{18}{24}$ and $\frac{20}{24}$: when added together, make $\frac{38}{24}$, or $1\frac{7}{12}$, which will also be true in the case of a pound sterling.

6thly, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, that is, $\frac{360}{720}$, $\frac{240}{720}$, $\frac{180}{720}$, $\frac{144}{720}$, $\frac{120}{720}$, when added together, make $\frac{1044}{720}$; that is, $1\frac{9}{20}$: try it in money.

7thly, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ and $\frac{5}{6}$, that is, $\frac{360}{720}$, $\frac{480}{720}$, $\frac{540}{720}$, $\frac{576}{720}$ and $\frac{600}{720}$, when added together, make $\frac{2556}{720}$, that is, $3\frac{11}{20}$.

8thly, The sum of the mixt numbers $7\frac{1}{3}$ and $8\frac{1}{4}$ is $15\frac{7}{12}$; for the sum of the fractions is $\frac{7}{12}$ by the second example, and the sum of the whole numbers is 15.

9thly, $5\frac{2}{3}$ added to $7\frac{4}{5}$ gives $13\frac{7}{15}$; for the sum of the fractions is $1\frac{7}{15}$ by the fourth example; and the whole number 1, added to the whole numbers 5 and 7 gives 13.

10thly,

10thly, $8\frac{1}{2}$, $9\frac{2}{3}$, $10\frac{3}{4}$, $11\frac{4}{5}$, $12\frac{5}{6}$ added together make $53\frac{11}{20}$; for the fractions themselves make $3\frac{11}{20}$ by the seventh example, and the whole number 3 added to the rest makes 53.

11thly, The whole number 2 added to the fraction $\frac{3}{4}$ gives $\frac{11}{4}$; for the whole number 2 may be considered as a fraction, whose denominator is unity; now $\frac{2}{1}$ and $\frac{3}{4}$ when reduced to the same denomination, are $\frac{8}{4}$ and $\frac{3}{4}$, which added together make $\frac{11}{4}$.

Thus also may unity be added to any fraction whatever, when subtraction requires it; but better thus: unity may be made a fraction of any denomination whatever, provided the numerator be equal to the denominator, by art. 2d: suppose then I would add unity to $\frac{2}{3}$; I suppose unity equal to $\frac{3}{3}$, and this added to $\frac{2}{3}$ makes $\frac{5}{3}$: again, unity added to $\frac{3}{5}$ makes $\frac{8}{5}$, because $\frac{5}{5}$ and $\frac{3}{5}$ make $\frac{8}{5}$.

Of the subtraction of fractions.

10. Whenever a less fraction is to be subtracted from a greater, they must be prepared as in addition; that is, they must be reduced to the same denomination, if they be not so already; then subtracting the numerator of the less fraction from that of the greater, put down the remainder with the common denominator under it. In the case of mixt numbers, subtract first the fraction of the lesser number from that of the greater, and then the lesser whole number from the greater: but if, as it often happens, the greater number has the lesser fraction belonging to it, then an unit must be borrowed from the whole number and added to the fraction, as intimated in the close of the last article.

Examples of subtraction in fractions.

1st, $\frac{3}{10}$ subtracted from $\frac{4}{10}$ leaves $\frac{1}{10}$, just in the same manner as 3 shillings subtracted from 4 shillings leaves 1 shilling.

2dly, $\frac{3}{4}$ subtracted from $\frac{5}{6}$, that is, $\frac{18}{24}$ subtracted from $\frac{20}{24}$, leaves $\frac{2}{24}$, or $\frac{1}{12}$. So $\frac{3}{4}$ of a pound, or 15 shillings, subtracted from $\frac{5}{6}$ of a pound, or 16 shillings and 8 pence, leaves $\frac{1}{12}$ of a pound, that is, 1 shilling and 8 pence.

3dly, $7\frac{1}{3}$ subtracted from $8\frac{1}{2}$, that is, $7\frac{2}{6}$ subtracted from $8\frac{3}{6}$, leaves $1\frac{1}{6}$.

4thly, $7\frac{3}{4}$ subtracted from $8\frac{1}{4}$, that is, $7\frac{3}{4}$ subtracted from $7\frac{5}{4}$, leaves $\frac{2}{4}$, or $\frac{1}{2}$; for here the greater number having the less fraction belonging to it, I borrow an unit from the whole number 8, and so reduce it to 7; and then this unit, under the name of $\frac{4}{4}$, I add to the fraction $\frac{1}{4}$, and so make it $\frac{5}{4}$.

5thly,

5thly, $7\frac{2}{3}$ subtracted from $8\frac{1}{2}$, that is, $7\frac{1}{2}$ subtracted from $8\frac{1}{2}$, that is, $7\frac{4}{6}$ subtracted from $7\frac{2}{3}$; leaves $\frac{2}{6}$.

6thly, $7\frac{1}{3}$ subtracted from 8, that is, $7\frac{2}{3}$ subtracted from $7\frac{2}{3}$, leaves $\frac{2}{3}$.

Of the multiplication of fractions.

11. To multiply a whole number is to take the multiplicand as often as that whole number expresses: therefore to multiply by a mixt number, is, not only to take the multiplicand as often as the integral part expresses, but also to take such a part or parts of it over and above, as is expressed by the fraction annexed. Thus 10 multiplied by $2\frac{1}{2}$, produces 25: for as $2\frac{1}{2}$ is a middle number between 2 and 3, so the product ought to be a middle number between 20 and 30, that is, 25: In like manner 10 multiplied by $1\frac{1}{2}$ produces 15, and being multiplied by $\frac{1}{2}$ produces 5: therefore to multiply by a proper fraction, is nothing else but to take such a part or parts of the multiplicand, as is expressed by that fraction. Certainly to take 10 twice and half of it over, once and half of it over, no times and half of it over, (which last is taking the half of 10,) are operations of the same kind, and differ only in degree one from another; and therefore, if the two former operations pass by the name of multiplication, this last ought to do so too; and if there be any absurdity in the case, it lies in the name, and not in the thing.

Arithmetic was at first employed about whole numbers only, and thus far the name of multiplication was adequate enough, except in the case of unity. But it being afterwards considered, that no quantity whatever could be called an unit, that was not further divisible; and consequently, that there was not only an infinity of fractional numbers below unity, but also an infinity of mixt numbers between any two whole numbers whatever; it was judged, rightly enough, that the art of Arithmetic would not be perfect till it's operations extended themselves to this sort of number also; and this being done without changing their names, it was then that the name of multiplication became too scanty for the thing signified: this therefore ought to be attributed to the unavoidable want of foresight in the first imposers, and not to any imperfection in the science itself. This is no more than the case of many other arts and sciences that have outgrown their names. Thus Geometry, that originally and properly signified no more than the art of surveying, is now defined to be a science treating of the nature and properties of all figures, or rather of the different modifications of extension and space; so that now surveying is the least and lowest part of that science. Thus Hydrostatics, which originally signified no more than the art of weighing bodies in water, or rather the art of finding out the specific gravities of
bodies

bodies by weighing them in water, is now made the name of a science, which treats of the nature and properties of fluids in general; and the several properties of air and mercury, so far as they are fluids, fall under the consideration of Hydrostatics, as properly as those of water.

But perhaps it may be further urged, that to take the half of any quantity, is not to multiply, but to divide it. To which I answer: that it is impossible to take the half of any quantity without dividing it by 2; and consequently, that to multiply by $\frac{1}{2}$ has the same effect as to divide by 2; but this does not prove that multiplication is the same as division, but only that these two operations, how contrary soever, may be made to do each others business, which is no mystery to any one who is the least conversant in Arithmetic, and will be further explained in the next article.

A fraction may be multiplied by a whole number two ways; either by multiplying the numerator by that number, or else by dividing the denominator by the same, where such a division is possible: thus if the fraction $\frac{2}{3}$ be to be multiplied by 2, the product will either be $\frac{4}{3}$ by doubling the numerator, or $\frac{2}{\frac{3}{2}}$ by halving the denominator: this is evident from the 6th art. because a fraction will be equally encreased, whether it be by encreasing the numerator, or by diminishing the denominator.

If a fraction be to be multiplied by a fraction, multiply the numerator and denominator of the multiplicand, by the numerator and denominator of the multiplier respectively, and the fraction thence arising will be the product sought: thus if it was required to multiply $\frac{2}{3}$ by $\frac{4}{5}$, or (which amounts to the same thing) if it was required to determine how much is $\frac{2}{3}$ of $\frac{4}{5}$, the answer would be $\frac{8}{15}$; and the reason is plain; for $\frac{1}{3}$ of $\frac{4}{5}$ is $\frac{4}{15}$, by the sixth art. because making the denominator three times greater, makes the fraction three times less; but if $\frac{1}{3}$ of $\frac{4}{5}$ be $\frac{4}{15}$, then $\frac{2}{3}$ of $\frac{4}{5}$ ought to be twice as much, that is, $\frac{8}{15}$; therefore to determine the amount of $\frac{2}{3}$ of $\frac{4}{5}$, the numerator and denominator of $\frac{2}{3}$ must be multiplied respectively by the numerator and denominator of $\frac{4}{5}$; and the same reason will hold good in all other instances.

If a whole number is to be multiplied by a fraction, either change the multiplier and multiplicand one for another, and then proceed as above directed; or else consider the multiplicand as a fraction whose denominator is unity, and so proceed according to the rule for multiplying one fraction by another; by which means both rules will be contracted into one. Thus 6, or $\frac{6}{1}$, multiplied into $\frac{2}{3}$, produces $\frac{12}{3}$, or 4.

If the multiplier, or multiplicand, or both, be mixt numbers, they must first be reduced to improper fractions by the third art. and then be multiplied according to the general rule.

Examples

Examples of multiplication in fractions.

1st, $\frac{2}{3}$ of $\frac{7}{8}$, multiplying numerators together, and denominators together, is $\frac{14}{24}$, or $\frac{7}{12}$: and so we find it in any particular case: for $\frac{7}{8}$ of a pound are 17 shillings and 6 pence; and $\frac{2}{3}$ of 17 shillings and 6 pence, that is, (by the 5th art.) $\frac{2}{3}$ of 35 shillings, is 11 shillings and 8 pence; therefore $\frac{2}{3}$ of $\frac{7}{8}$ of a pound are 11 shillings and 8 pence, which will also be found to be the value of $\frac{7}{12}$ of a pound.

Here we may observe once for all, that whenever two fractions are to be multiplied together, the product will be the same, which soever it is that multiplies the other, just as it is in whole numbers, and for the same reason; for if $\frac{7}{8}$ be to be multiplied by $\frac{2}{3}$, then the numbers 7 and 8 must be respectively multiplied by 2 and 3; but if $\frac{2}{3}$ is to be multiplied by $\frac{7}{8}$, then the numbers 2 and 3 must be respectively multiplied by 7 and 8, which amounts to the same thing; whence it follows, that $\frac{2}{3}$ of $\frac{7}{8}$ come to the same as $\frac{7}{8}$ of $\frac{2}{3}$: to confirm this, we have seen already that $\frac{2}{3}$ of $\frac{7}{8}$ of a pound amount to 11 shillings and 8 pence; let us in the next place enquire into the value of $\frac{7}{8}$ of $\frac{2}{3}$ of a pound: now $\frac{2}{3}$ of a pound are 13 shillings and 4 pence; and $\frac{7}{8}$ of 13 shillings and 4 pence, that is, $\frac{1}{8}$ of 93 shillings and 4 pence, is 11 shillings and 8 pence; therefore $\frac{2}{3}$ of $\frac{7}{8}$ of a pound are the same as $\frac{7}{8}$ of $\frac{2}{3}$ of a pound, since both amount to 11 shillings and 8 pence.

2^{dly}, $\frac{2}{3}$ of $\frac{9}{10}$ are $\frac{90}{180}$, or $\frac{1}{2}$: for $2 \times 5 \times 9$ make 90, and $3 \times 6 \times 10$ make 180: thus $\frac{9}{10}$ of a pound are 18 shillings; and $\frac{2}{3}$ of 18 shillings are 12 shillings; and $\frac{2}{3}$ of 12 shillings are 8 shillings; which are $\frac{1}{2}$ of a pound.

3^{dly}, $\frac{3}{4}$ of $\frac{3}{4}$ are $\frac{27}{64}$: thus $\frac{3}{4}$ a pound are 15 shillings; and $\frac{3}{4}$ of 15 shillings are 11 shillings and 3 pence; and $\frac{3}{4}$ of 11 shillings and 3 pence are 8 shillings and 5 pence farthing; which will also be found to be the value of $\frac{27}{64}$ of a pound.

4^{thly}, The mixt number $6\frac{3}{4}$ multiplied by the whole number 7, or the whole number 7 multiplied by the mixt number $6\frac{3}{4}$, will produce in either case $47\frac{1}{4}$: for the mixt number $6\frac{3}{4}$ being reduced (by the 3d art.) to an improper fraction, becomes $\frac{27}{4}$; which being multiplied by 7, or $\frac{7}{1}$, makes $\frac{189}{4}$, or, when reduced to a mixt number, $47\frac{1}{4}$.

This multiplication may also be made another way, thus: $\frac{3}{4}$ multiplied by 7 makes $\frac{21}{4}$, that is, (by the 2d art.) $5\frac{1}{4}$; put down the fraction $\frac{1}{4}$ and keep the 5 in reserve; then 6 multiplied by 7 makes 42, which,

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with the 5 in reserve, makes 47; therefore the whole product is $47\frac{1}{4}$ as before.

5thly, $3\frac{3}{4}$ multiplied by $2\frac{2}{3}$, that is, $\frac{15}{4}$ multiplied by $\frac{8}{3}$, makes $\frac{120}{12}$, that is, 10: thus $3\frac{3}{4}$ of a pound are 3 pounds, 15 shillings; and twice 3 pounds, 15 shillings is 7 pounds, 10 shillings; moreover $\frac{2}{3}$ of 3 pounds, 15 shillings, or $\frac{1}{3}$ of 7 pounds, 10 shillings, is 2 pounds, 10 shillings; and these 2 pounds, 10 shillings added to the former part of the product, to wit, 7 pounds, 10 shillings, give 10 pounds for the whole product; therefore $3\frac{3}{4}$ of a pound multiplied by $2\frac{2}{3}$ make 10 pounds.

6thly, $96\frac{1}{2}$ multiplied by $24\frac{1}{3}$, that is, $\frac{193}{2}$ multiplied by $\frac{73}{3}$, gives $\frac{14089}{6}$, that is, (by the 2d art.) $2348\frac{5}{6}$.

7thly, $36\frac{1}{4}$ multiplied into itself, that is, $\frac{145}{4}$ multiplied by $\frac{145}{4}$ makes $\frac{21025}{16}$, that is, $1314\frac{1}{16}$.

Before I put an end to this art. I do not know whether it will be thought worth my while to take notice of a very absurd question sometimes bandied about, wherein it is required to multiply $\frac{1}{3}$ of a pound by $\frac{1}{2}$ of a pound: I call this a very absurd question, because there is no manner of propriety in it; for in the very idea and definition of multiplication, the multiplicator at least is supposed to be an abstract number, or fraction, otherwise, what can be the meaning of taking the multiplicand as often, or as much of it, as is expressed by the multiplicator? If by multiplying $\frac{1}{3}$ of a pound by $\frac{1}{2}$ of a pound, be meant no more, than multiplying $\frac{1}{3}$ of a pound by $\frac{1}{2}$, why is the word pound expressed in the multiplicator? and if there be any other meaning in it, why does not the proposer explain it, since it is not expressed in the question? Let him tell me what he means by multiplying 1 pound by 1 pound, and I will soon undertake to answer his question; but if he neither can nor will do this, the question neither deserves, nor is capable of an answer. I am not ignorant of another question more frequently used than this, and of equal nonsense, if custom had not explained it, and that is, to multiply 3 yards by 2 yards, and the like; whereby is meant I suppose, to assign the number of square yards contained in a rectangled parallelogram, or long square, 3 yards in length, and 2 yards in breadth; but if this be the sense put upon that question by common consent, that is all the title it has to it, there being no such thing either expressed, or so much as implied, in the terms of the question.

A L E M M A.

12. *Let n be any whole number, mixt number, or fraction; I say then that the quotient of n divided by any fraction is equal to the product of n multiplied into the reverse of that fraction: as for instance,*

Let n be divided by $\frac{1}{2}$; I say that the quotient of n divided by $\frac{1}{2}$ will be equal to the product of n multiplied by $\frac{2}{1}$: for let q be the quotient of n divided by $\frac{1}{2}$; that is, let q be a number expressing how often the fraction $\frac{1}{2}$ is contained in n ; then will $\frac{1}{2}$ multiplied by q , be equal to n , from the nature of multiplication; but the product of $\frac{1}{2}$ multiplied by q is the same with the product of q multiplied by $\frac{1}{2}$; that is, $\frac{1}{2}$ of q , by the last article; therefore n is equal to $\frac{1}{2}$ of q ; therefore $\frac{1}{2}$ of n is equal to $\frac{1}{4}$ of q ; therefore $\frac{2}{1}$ of n are equal to q ; but $\frac{2}{1}$ of n is the product of n multiplied by $\frac{2}{1}$; therefore the product of n multiplied by $\frac{2}{1}$ is equal to q ; but the quotient of n divided by $\frac{1}{2}$ was q , by the supposition; therefore the quotient of n divided by $\frac{1}{2}$ is equal to the product of n multiplied by $\frac{2}{1}$.

Q. E. D.

C O R O L L A R Y.

Hence may the rule of division be at any time changed into that of multiplication, only by inverting the terms of the divisor, and then multiplying instead of dividing. The same will also obtain in whole numbers, if they be considered as fractions whose denominators are units: thus to divide n by 2, that is, $\frac{2}{1}$, will have the same effect as to multiply it by $\frac{1}{2}$, as was hinted in the foregoing article.

Of the division of fractions.

13. The division of fractions like all other division, is to find how often one fraction called the divisor, is contained in another called the dividend; and that which shews this, is called the quotient, whether it be a whole number, a mixt number, or a proper fraction: for in fractional division the quotient is always intended to be exact, without any remainder, and therefore must sometimes be a whole number, sometimes a mixt number, and sometimes a proper fraction. Thus if 18 is to be divided by 6, the quotient will be 3; because 18 contains 6 3 times: but if 21 is to be divided by 6, the quotient will be $3\frac{1}{2}$; because 21 contains 6 three times, and half of it over and above: lastly, if $\frac{3}{2}$ is to be divided by $\frac{6}{1}$, the quotient will be $\frac{1}{4}$; because here the divisor being greater

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than

than the dividend, cannot be so much as once contained in it, and therefore the quotient in this case, must be a proper fraction, that is, $\frac{1}{2}$, since 3 is just the half of 6.

A fraction may be divided by a whole number two ways; either by dividing the numerator by that whole number when possible, or else by multiplying the denominator by the same: thus the half of $\frac{6}{7}$ may be taken, that is, $\frac{6}{7}$ may be divided by 2, either by halving the numerator, and the quotient will be $\frac{3}{7}$, or else by doubling the denominator, and then the quotient will be $\frac{6}{14}$, both which amount to the same thing, by the 6th and 7th articles.

If the divisor be a fraction, the quotient may be had by multiplying the dividend into the inverted divisor, according to the rules of multiplication already laid down: thus if $\frac{4}{5}$ is to be divided by $\frac{2}{3}$, the quotient will be the same as the product of $\frac{4}{5}$ multiplied by $\frac{3}{2}$, that is, $\frac{12}{10}$, or $1\frac{1}{5}$; the demonstration whereof is contained in the last article.

And here again, as well as in the eleventh article, we are to observe, that if either the divisor, or dividend, or both, be mixt numbers, they must be reduced to improper fractions before the general rule can have place; and that if either, or both be whole numbers, they must be considered as fractions whose denominators are units.

From the general rule of division before laid down it follows, that every fraction may be considered as the quotient of the numerator divided by the denominator, and that, whether the terms of the fraction under consideration be whole numbers, or (which sometimes happens) mixt numbers, or even pure fractions: a demonstration of this last case will serve for all, since mixt numbers may be reduced to fractions, and whole numbers may be considered as fractions whose denominators are units.

Let the fraction proposed be $\frac{4}{5}$; I say, that this fraction is equal to the quotient arising from the division of the numerator $\frac{4}{5}$ by the denominator $\frac{2}{3}$: to demonstrate which, multiply both $\frac{4}{5}$ the numerator, and $\frac{2}{3}$ the denominator, by $\frac{3}{2}$ the inverted denominator, and the fraction will be changed into this, $\frac{12}{10}$ or $\frac{12}{10}$, being of the same value with the former, by the 6th art.

but the quotient of $\frac{4}{5}$ divided by $\frac{2}{3}$ is also $\frac{12}{10}$ as above; therefore the fraction $\frac{4}{5}$ is equal to the quotient arising from the division of the numerator by the denominator: and the same way of reasoning may be used in any other instance. This consideration is of very great use in Algebra, where quantities are very often so generally expressed, that there is no other way of representing the quotient, but by a fraction whose numerator is the dividend, and denominator the divisor. Hence also we are taught
how

how to reduce a complicated fraction into a simple one, whose numerator and denominator are whole numbers, to wit, by dividing the numerator by the denominator: thus we see that $\frac{\frac{4}{5}}{\frac{2}{3}}$ is the same as $\frac{12}{10}$.

Other examples of division in fractions.

1st, $\frac{5}{6}$ divided by $\frac{3}{4}$, or, which is the same thing, $\frac{5}{6}$ multiplied into $\frac{4}{3}$, makes $\frac{20}{18}$, or $1\frac{2}{9}$; which shews, that $\frac{3}{4}$ is contained once, and $\frac{2}{9}$ part of it over and above, in $\frac{5}{6}$: for a further confirmation of this, $\frac{5}{6}$ of a pound are 16 shillings and 8 pence; and $\frac{3}{4}$ of a pound are 15 shillings: now 15 shillings are once contained in 16 shillings and 8 pence, and there is 1 shilling and 8 pence over; which 1 shilling and 8 pence is just $\frac{2}{9}$ of 15 shillings. To prevent oversights, the learner is to remember, that it is the terms of the divisor only that are to be inverted, and not those of the dividend: thus to divide $\frac{5}{6}$ by $\frac{3}{4}$ is the same as to multiply $\frac{5}{6}$ into $\frac{4}{3}$, but not the same as to multiply $\frac{6}{5}$ into $\frac{4}{3}$.

2^{dly}, $\frac{9}{10}$ divided by $\frac{1}{3}$, or multiplied into $\frac{3}{1}$, make $\frac{27}{10}$, or $2\frac{7}{10}$, which may be confirmed like the former: for $\frac{9}{10}$ of a pound are 18 shillings; and $\frac{1}{3}$ of a pound is 6 shillings and 8 pence: now 6 shillings and 8 pence are twice contained in 18 shillings, and there are 4 shillings and 8 pence over; which 4 shillings and 8 pence will be found by the 5th art. to be just $\frac{7}{10}$ of 6 shillings and 8 pence.

3^{dly}, The whole number 10 divided by $2\frac{2}{3}$, that is $\frac{10}{1}$ divided by $\frac{8}{3}$, or multiplied into $\frac{3}{8}$, makes $\frac{30}{8}$, or $3\frac{3}{4}$.

4^{thly}, $2\frac{2}{3}$ divided by $\frac{10}{12}$, or $\frac{8}{3}$ divided by $\frac{5}{6}$, or multiplied into $\frac{6}{5}$ makes $\frac{8}{5}$, or $1\frac{3}{5}$.

5^{thly}, $16\frac{1}{3}$ divided by $1\frac{1}{6}$, that is, $\frac{49}{3}$ divided by $\frac{7}{6}$ or multiplied into $\frac{6}{7}$ makes $\frac{294}{21}$, or 14.

Further observations concerning multiplication and division in fractions.

14. When two fractions are multiplied together, or one is divided by the other, it often happens, that though the original fractions be both in their least terms, yet the product, or quotient from them, shall be otherwise, and require a further reduction: as for instance, the fractions $\frac{2}{5}$ and $\frac{9}{10}$ are both in their least terms; and yet if they be multiplied together, their product $\frac{18}{50}$ is so far from being in it's least terms, that it may be reduced to $\frac{9}{25}$: so again in division, $\frac{9}{10}$ and $\frac{15}{16}$ are fractions both

in their least terms; and yet if the latter be divided by the former, the quotient $\frac{150}{144}$ is reducible to $\frac{25}{24}$. It may not be amiss therefore, to enquire into the cause of this, and see whether the original fractions may not be so prepared beforehand, as that the product, or quotient, shall always come out in it's least terms. First then, as to the multiplication of $\frac{5}{6}$ and $\frac{2}{10}$; here it is easy to see, that the product of $\frac{5}{6}$ and $\frac{2}{10}$ multiplied together, will just amount to the same, as that of $\frac{5}{10}$ into $\frac{2}{6}$, the denominators of the fractions being interchanged; this I say is certain from the operation itself; for the same numbers are multiplied together in both cases; but these last fractions are far from being in their least terms, the former, $\frac{5}{10}$ being reducible to $\frac{1}{2}$, and the latter $\frac{2}{6}$ to $\frac{1}{3}$; but after these new fractions $\frac{1}{2}$ and $\frac{1}{3}$ are reduced to their least terms $\frac{1}{2}$ and $\frac{1}{3}$; their product $\frac{1}{6}$ will be the same in value with that of the original fractions, and at the same time will be in it's least terms. Thus then we see, that to have the product in it's least terms, care must be taken, not only to reduce the original fractions as low as possible, but after that, to interchange their denominators, and then again to reduce these new fractions to their least terms, and lastly, to multiply these reduced fractions one into another.

The same manner of practice will also serve for division, after it is reduced to the rule of multiplication: as for example; the quotient of $\frac{15}{16}$ divided by $\frac{2}{10}$, is the same with the product of $\frac{15}{16}$ multiplied into $\frac{10}{2}$; and this again, is the same with the product of $\frac{15}{2}$ multiplied into $\frac{10}{16}$, as above; but because the fractions $\frac{15}{2}$ and $\frac{10}{16}$, are not in their lowest terms, they must be reduced to $\frac{15}{2}$ and $\frac{5}{8}$ before it can be expected that their product $\frac{25}{24}$ should be in it's least terms. Thus we have reduced the two compendiums of multiplication and division, not only to one rule instead of two, as they are commonly given out, but also to such a rule as carries it's own evidence along with it.

N. B. What was here done by interchanging the denominators, and keeping the numerators in their places, may as well be done by interchanging the numerators, and keeping the denominators in their places, the reason of both being the same.

Of the rule of proportion in fractions.

15. The rule of proportion in fractions, is so much the same with the rule of proportion in whole numbers, that nothing more needs to be said of it, except to illustrate it by an example or two.

Examples

Examples of the rule of proportion in fractions.

1st, If $\frac{1}{4}$ give $\frac{1}{5}$, what will $\frac{1}{6}$ give? Here $\frac{1}{5}$ and $\frac{1}{6}$ multiplied together give $\frac{1}{30}$; and this divided by $\frac{1}{4}$ (or multiplied by $\frac{4}{1}$) quotes $\frac{4}{30}$, or $\frac{2}{15}$, which is an answer to the question.

2dly, If $2\frac{2}{3}$ give $3\frac{3}{4}$, what will $4\frac{4}{5}$ give? These mixt numbers being by the 3d art. reduced to improper fractions, will stand thus: If $\frac{8}{3}$ give $\frac{15}{4}$, what will $\frac{24}{5}$ give? Here $\frac{15}{4}$ and $\frac{24}{5}$ multiplied together give $\frac{360}{20}$ or 18; and this divided by $\frac{8}{3}$, quotes $6\frac{3}{4}$, which is an answer to the question.

3dly, If $\frac{1}{2}$ of a yard cost $\frac{1}{3}$ of a pound, what will $\frac{1}{4}$ of an ell cost? Here it must be observed, that an ell is $\frac{5}{4}$ of a yard, and consequently that $\frac{1}{4}$ of an ell is $\frac{1}{4}$ of $\frac{5}{4}$, or $\frac{5}{16}$ of a yard; so that the question may be stated thus: If $\frac{1}{2}$ of a yard cost $\frac{1}{3}$ of a pound, what will $\frac{5}{16}$ of a yard cost? Here $\frac{1}{3}$ and $\frac{5}{16}$ multiplied together make $\frac{5}{48}$, and this divided by $\frac{1}{2}$ quotes $\frac{5}{24}$ of a pound, or 4 shillings and twopence; which therefore is an answer to the question.

The reduction of proportion from fractional to integral terms.

Whenever two fractions are proposed, as $\frac{2}{3}$ and $\frac{4}{5}$, whose proportion is defined in whole numbers, reduce the fractions first to the same denomination by the 8th art. that is, in the present case, to $\frac{10}{15}$ and $\frac{12}{15}$; then you will have $\frac{2}{3}$ to $\frac{4}{5}$ as $\frac{10}{15}$ is to $\frac{12}{15}$; but $\frac{10}{15}$ is to $\frac{12}{15}$ as 10 to 12, or as 5 to 6; therefore $\frac{2}{3}$ is to $\frac{4}{5}$ as 5 to 6: here we may observe, that though the finding of the common denominator be necessary for understanding the reason of the rule, yet it is not at all necessary for the practice of it; for to what purpose is it to find the common denominator, to throw it away again when we have done? In practice therefore, multiply the numerator of the fraction which is the first in the proportion, by the denominator of the second, and then the numerator of the second fraction, by the denominator of the first, and the two products will exhibit respectively, the proportion of the first fraction to the second in whole numbers, as was evident in the foregoing example.

Of the extraction of roots in fractions.

16. As every fraction is squared, or multiplied into itself, by squaring both the numerator and denominator, (see art. 11.) so *e converso* the square

square root of every fraction will be obtained by extracting the square root both of the numerator and denominator: thus the square of $\frac{1}{4}$ is $\frac{1}{16}$, and the square root of $\frac{9}{16}$ is $\frac{3}{4}$. But here care must be taken, whenever the square root of a fraction is to be extracted, that the fraction itself be first reduced to it's simplest terms, by the 7th art. otherwise the fraction may admit of a square root, and yet this root may not be discovered: thus if it was required to extract the square root of the fraction $\frac{18}{32}$, it would be impossible to obtain the root either of 18 or 32; and yet when this fraction is reduced to it's least terms $\frac{9}{16}$, it's square root will be found to be $\frac{3}{4}$.

When the square root of a number cannot be extracted exactly, it is usual to make an approximation by the help of decimals, or otherwise, and so to approach as near to the value of the true root as occasion requires. Now in the case of a fraction, if the square root of neither the numerator nor denominator can be exactly obtained, there will be no necessity however, for two approximations, because such a fraction may be easily reduced to another of the same value, whose denominator is a known square: as for instance; suppose the square root of $46\frac{1}{5}$, or $\frac{231}{5}$ was required: I multiply both the numerator and denominator of this fraction by 5, and so reduce it to $\frac{1155}{25}$: Here the denominator 25 is a known square number, whose root is 5; and the square root 1155 is 34 nearly; therefore, the square root of the fraction proposed is nearly $\frac{34}{5}$, or $6\frac{4}{5}$. But after all, the best way of extracting the square root of a vulgar fraction, is by throwing it into a decimal fraction, as will be shewn hereafter.

Note. That whatever has here been said concerning the extraction of the square root in fractions, may easily be applied, *mutatis mutandis*, to the extraction of the cube root, &c.

Of decimal fractions.

And first of their notation.

17. A decimal fraction is a fraction whose denominator is 10, or 100, or 1000, or 10000, &c. and this denominator is never expressed, but always understood by the place of the figure it belongs to: for as all figures on the left hand of the place of units, rise in their value, according to their distances from it, in a decuple proportion; so all figures on the right hand of the place of units, sink in their value in a subdecuple proportion: as for instance; the number 345.6789, where 5 stands in

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in the place of units, is to be read thus; *three hundred forty five, six tenths, seven hundredth parts, eight thousandth parts, nine tenthousandth parts*: or the decimal parts may be read thus; *six thousand seven hundred eighty nine tenthousandth parts*; the denominator being ten thousand, because the last figure 9, according to the former way of reckoning, stands in the place of tenthousandth parts. The reason of this latter way of reading is plain; for $\frac{6}{10}$ are $\frac{6000}{10000}$, and $\frac{7}{100}$ are $\frac{700}{10000}$, and $\frac{8}{1000}$ are $\frac{80}{10000}$, and $\frac{6000}{10000}$, $\frac{700}{10000}$, $\frac{80}{10000}$ and $\frac{9}{10000}$, all added together, make $\frac{6789}{10000}$.

Cyphers are used in the expression of decimals as well as whole numbers, and for the same reason. Thus .067 may be read either *no tenths, six hundredth parts, seven thousandth parts*; or *sixty seven thousandth parts*. But cyphers on the right hand of a decimal number (if nothing follows them) are as insignificant as cyphers on the left hand of a whole number; and yet cyphers are sometimes placed after decimals, for the sake of regularity, or when we want to increase the number of decimal places.

From what has here been said, it will be easy to multiply or divide any number by 10, 100, 1000, &c only by removing the separating point towards the right or left hand. Thus the number 345.6789 being multiplied by 10, becomes 3456.789; and being multiplied by 100, becomes 34567.89; and the same number 345.6789 being divided by 10, becomes 34.56789; and being divided by 100, becomes 3.456789; thus again, the number 345 being divided by 10000, becomes .0345; for to divide by 10000, is the same thing as to remove the separating point 4 degrees towards the left hand, if there be any separating point in the number given; but if there be none, as in the present case, then to put a separating point four degrees towards the left hand, which in this example cannot be done, but by the help of a cypher in the first decimal place.

Of the addition and subtraction of decimal fractions.

18. The chief advantage of decimal arithmetic above that of common fractions, consists in this, that in decimals, all operations are performed as in whole numbers: this will presently appear from the several parts of decimal arithmetic, as they come now to be treated of in order; and first of addition and subtraction.

Addition and subtraction in decimals are performed after the same manner as in whole numbers, care being taken, that like parts be placed under

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der one another : as for example, .567 are added to .89 thus ;

$$\begin{array}{r}
 .89 \\
 .567 \\
 \hline
 1.457
 \end{array}
 \quad
 \begin{array}{l}
 \text{subtracted thus;} \\
 \end{array}
 \quad
 \begin{array}{r}
 .89 \\
 .567 \\
 \hline
 .323
 \end{array}
 \quad
 \begin{array}{l}
 \text{or thus;} \\
 \end{array}
 \quad
 \begin{array}{r}
 .890 \\
 .567 \\
 \hline
 .323
 \end{array}$$

Of the multiplication of decimal fractions.

19. Multiplication in decimals is also performed as in whole numbers, no regard being had to the decimals as such, till the product is obtained ; but then, so many decimal places must be cut off from the right hand of the product, as are contained both in the multiplicator and multiplicand : as for instance ; let it be required to multiply 4 .56 by 2 .3 : here considering both factors as whole numbers, I multiply 456 by 23, and find the product to be 10488 ; but then considering that there was one decimal in the multiplicator, and two in the multiplicand, I cut off three decimal places from the right hand of the product, and the true product stands thus ; 10 .488.

To shew the reason of this operation, let the two factors be reduced to simple fractions according to the common way, and we shall have 2 .3 equal to $\frac{23}{100}$, and 4 .56 equal to $\frac{456}{100}$, and these two fractions multiplied together make $\frac{10488}{10000}$; divide by 1000, which is done by cutting off the three last figures, according to art. the 17th, and the quotient will be 10 .488. Another example may be this : let it be required to multiply 45600 by .23 : the product of 45600 multiplied by 23 is 1048800 : but as there were two decimals in the given multiplicator, and none in the multiplicand ; I cut off two decimal places from the last product, and the true product will be found to be 10488 .00, or 10488. Lastly, let it be required to multiply .000456 by .23 : here neglecting the initial cyphers in the multiplicand, I multiply 456 by 23, and the product is 10488 : then I consider, that there were two decimal places in the multiplicator, and six in the multiplicand, and consequently that eight decimal places are to be cut off from the last product : but the last product consists of only 5 places ; therefore I place three cyphers to the left hand, with the separating point before them, and so make the true product .00010488.

There are various compendiums of this sort of multiplication to be met with in *Oughtred* and others ; but they are such, as by a little exercise, any one tolerably well grounded in this part of Arithmetic will easily discover of himself as they lie in his way.

Of the division of decimal fractions.

20. Division in decimal fractions is performed, first by considering them as whole numbers, and dividing accordingly; and then cutting off from the right hand of the quotient, as many decimal places as the dividend hath more than the divisor. The reason whereof is manifest from the 19th article: for since the divisor and quotient multiplied together are to make the dividend, the divisor and quotient ought to have as many decimal places between them, as there are in the dividend; therefore the quotient alone ought to have as many decimal places as the dividend hath more than the divisor.

Example the 1st; Let it be proposed to divide 10.488 by 2.3: here dividing the whole number 10488 by the whole number 23, I find the quotient to be 456: but then considering that there were 3 decimal places in the dividend, and but one in the divisor, I cut off two places from the right hand of the quotient, and so make the true quotient 4.56.

Example 2^d; Let it be proposed to divide 5678.9 by .06: here because there are two decimal places in the divisor, and but one in the dividend, I supply the deficient place by putting a cypher after the dividend, thus, 5678.90; then dividing the whole number 567890, by the whole number 6, (for since 6 is now considered as a whole number, the cypher before it may be neglected;) I find the quotient to be 94648, which is not to be sunk, because the dividend was made to have as many decimal places as the divisor; but as this quotient is not exact, if for a greater degree of exactness I would continue it to any number of decimal places, suppose 2, instead of one cypher after the divisor, I would have put three, and then the quotient would have come out 94648.33, and this quotient is much more exact than the former, as lying between 94648.33 and 94648.34: but it ought further to be observed concerning this quotient, that if the division was to be continued *in infinitum*, the figures in the decimal places would be all 3's: this is evident from the work; for the two last dividuums are the same, and therefore they must all be the same.

To reduce a vulgar fraction to a decimal fraction.

21. Since every fraction may be considered as the quotient of the numerator divided by the denominator, (see art. 13th,) we have an easy rule for reducing a vulgar fraction to a decimal fraction, which is as follows: put as many cyphers after the numerator, as are equal in

number, to the number of decimal places whereof you intend your reduced fraction to consist, and call these cyphers decimal; and then dividing the numerator by the denominator, the quotient will be a decimal number equal to the fraction first proposed, or perhaps a mixt number, if the fraction proposed was an improper one.

Example 1st; Let this fraction $\frac{3}{49}$ be proposed to be reduced to a decimal one consisting of four decimal places: here putting 4 decimal cyphers after the numerator 3, I divide 3.0000 by 49, and the quotient uncorrected is 612: but now considering that there were 4 decimal places in the dividend, and none in the divisor, and consequently that four decimal places are to be cut off from the quotient, whereas it consists but of three; I supply this defect of places by a cypher at the left hand, and so make the quotient .0612.

Example 2^d; Let this fraction $\frac{7}{16}$ be proposed to be reduced to a decimal fraction consisting, if possible, of six places: here dividing 7.000000 by 16, I find the true quotient to be .4375 the two last cyphers in the dividend being useless.

Note. When this division runs *ad infinitum*, it will be impossible for the reduction to be exact in a finite number of terms; but an approximation may be made that shall come nearer to the quotient than the least assignable difference, by taking more and more terms.

To reduce the decimal parts of any integer to such other parts as that integer is usually divided into.

22. To explain this rule, and to give an example of it at the same time; let .345 of a pound sterling, that is, three hundred forty five thousandth parts of a pound be given to be reduced into shillings, pence and farthings: here then I observe, that as any number of pounds multiplied by 20 will give as many shillings as are equal to the pounds, so any decimal parts of a pound multiplied by 20, will give as many shillings, and decimal parts of a shilling, as are equivalent to the decimal parts of a pound; and so on as to pence and farthings: multiplying therefore .345 by 20, the product is 6 and .900, or 6.9, which signifies, that .345 of a pound are equivalent to six shillings and nine tenths of a shilling, which is usually written thus; 6.9 shillings: again, multiplying this last decimal .9 by 12 for pence, I find that .9 of a shilling are equivalent to 10.8 pence: lastly, multiplying .8 by 4 for farthings, I find that .8 of a penny are equivalent to 3.2 farthings; as for the .2 of a farthing, I neglect it, there being no lower denomination,

or

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or at least, not intending to descend any lower; and so I find .345 of a pound to amount to six shillings and tenpence three farthings.

To reduce the common parts of any integer into equivalent decimal parts of the same.

23. This reduction being the reverse of the former, it might be performed by division, as that was by multiplication; but when all things are considered, I do not know whether the following method may not be thought as easy, and as intelligible as any: let it then be required to reduce 2 hours, 34 minutes, 56 seconds, into equivalent decimal parts of a day. Now in one day there are 86400 seconds; and in two hours, 34 minutes, 56 seconds, there are 9296 seconds; therefore two hours, 34 minutes, 56 seconds, are equivalent to $\frac{9296}{86400}$ of one day: reduce this vulgar fraction to an equivalent decimal, by the last article but one, and you will find it to be .10759; therefore 2 hours, 34 minutes, 56 seconds, are equivalent to .10759 of one day. But there is one article still remains to be adjusted, and that is, to how many decimal places the foregoing fraction must be reduced, so as to express accurately enough, the parts of a day to a second of time. Now to know this, I consider that one second of time is $\frac{1}{86400}$ of one day; therefore I reduce $\frac{1}{86400}$ to a decimal fraction, at least as far as to the first significant figure, and find it to be .00001; whence I conclude, that to express the parts of a day to a second of time by any decimal, that decimal must not consist of fewer than 5 places, because there were 5 places in the decimal fraction .00001. Now to shew that the decimal fraction above found, to wit, .10759 expresses the time proposed to a second, reduce it back again, by the last art. and you will find it amount to 2 hours, 34 minutes, 55.8 seconds.

For another example, let us take the reverse of that in the last art. that is, let it be proposed to reduce six shillings ten pence 3.2 farthings, into equivalent decimal parts of a pound: one pound contains 960 farthings, or 9600 tenths of a farthing; and 6 shillings, 10 pence 3.2 farthings, contain 3312 tenths of a farthing; therefore 6 shillings, 10 pence 3.2 farthings, are equivalent to $\frac{3312}{9600}$ of a pound; but $\frac{1}{9600}$ being reduced to a decimal, is .0001 &c. wherein the first significant figure is in the 4th place; therefore I reduce the fraction $\frac{3312}{9600}$ to four decimal places, and they amount to .3450, that is, .345 of a pound; so that
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in this particular case, three decimal places are sufficient to express exactly the sum proposed.

Of the extraction of the square root in decimal fractions.

24. Having treated of the multiplication and division of decimal fractions, it would be altogether needless to say any thing concerning the rule of proportion, which is but a particular application of both: therefore I shall now pass on to the extraction of the square root, at least so far as it concerns decimal fractions. There are but few square numbers, or such as will admit of an exact square root, in comparison of the rest; and therefore, whenever a number is proposed to have its square root extracted, the artist must first determine with himself, to how many decimal places it is proper the root should be continued; and then by annexing decimal cyphers, if need be, to the right hand of the number proposed, he must make twice as many decimal places there, as the root is to consist of; after this, he must put a point over the place of units, and then passing by every other figure, he must point in like manner all the rest, both to the right hand, and to the left: by this means, the number will be prepared, and the square root may be extracted as in whole numbers, provided that so many decimal places be cut off from the root when obtained, as were first designed.

Example 1st; Let the root of 2345.6 be required to two decimal places. The number when prepared, stands thus, 2345.6000 , or as a whole number, thus, 23456000 ; and its square root, when extracted, will be 4843 nearly; and therefore 48.43 will be the root sought. To try this root 48.43, multiply it into itself, and the 4 first figures of the square will be 2345, which are all true, nor can it be expected any more should be so, because there were but four places true in the root, no notice being taken of the rest: but had the root been extracted true to 5 places, that is, to as many places as the original square consisted of, it would then have been 48.431; multiply this number into itself, and 5 of the first figures of the product, taken with the least error, will be 2345.6, which is the original square itself.

Example 2^d; Let the root of .0023456 be required to 5 decimal places. Here putting a cypher in the place of units to direct the punctuation, thus, 0.0023456000 , I extract the square root of 23456000 as of a whole number, and find it to be 4843, as above: but considering

that

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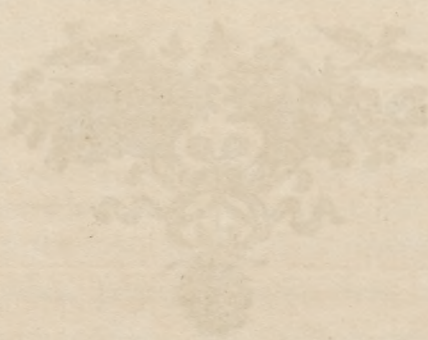
that this root is to be sunk 5 places, I put a cypher to the left hand, and so make the true root .04843.

That the supposed square ought to have twice as many decimal places as the root, is evident, both *à priori*, and *à posteriori*: *à priori*, because in extracting the square root, two figures are brought down from the square for every single figure gained in the root; and *à posteriori*, because the root multiplied into itself is to produce the square; and therefore, from the nature of multiplication, the square ought to have twice as many decimal places as the root.



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that this root is to be taken a place, I put a cipher to the left hand, and to make the true root .0444.
 That the supposed square ought to have twice as many decimal places as the root, is evident, both 4 times, and 4 figures; because in extracting the square root, two figures are brought down from the square for every single figure found in the root; and 4 figures; because the root multiplied into itself is to produce the square; and therefore from the nature of multiplication, the square ought to have twice as many decimal places as the root.



T H E

ELEMENTS of ALGEBRA

B O O K I.

The Definition of Algebra.

Article 1. **I** SHALL not here detain the young student with a long historical account of the rise and progress of Algebra; nor even so much as with either the etymology or signification of the word, which would contribute but very little to his information, till he has made a further progress in the science itself, and whereof he will find enough in Dr. *Wallis* and others. Nor indeed is it a subject altogether so proper at this time to be insisted upon; this art, like many others, having now considerably outgrown its name, and being often employed in arithmetical operations very different from what its name imports. All I shall advance then, by way of definition is, that *Algebra*, in the modern sense of the word, *is the art of computing by symbols*, that is, generally speaking, by letters of the alphabet; which for the simplicity and distinctness both of their sounds and characters, are much more commodious for this purpose than any other symbols or marks whatever.

In this way of notation, it is usual to substitute letters not only for such quantities as are unknown, and consequently, such as cannot well be represented otherwise, but also for known quantities themselves, in order to keep them distinct one from another, and to form general conclusions. As for instance; suppose it was demanded of me, what two numbers are those, whose sum is 48, and whose difference is 14: here, if I only put x , or some other letter for one of the unknown quantities, and use the known ones 48 and 14 as I find them in the problem, I shall

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only come to this particular conclusion, to wit, that the greater number is 31, and the less 17, which numbers will answer both the conditions of the problem. But if instead of the known numbers 48 and 14, I substitute the general quantities a and b respectively, and so propose the problem thus; *What two numbers are those, whose sum is a , and whose difference is b ?* I shall then come to this general conclusion, viz. that *Half the sum of a and b will be the greater number, and half their difference will be the less*: which general theorem will suit not only the particular case abovementioned, but also all other cases of this problem that can possibly be proposed. How I come by these two conclusions, will be sufficiently shewn in the course of this work; as also many other advantages attending this way of substituting letters for known quantities, besides those already mentioned.

What I have here said, was only to illustrate in some measure, the definition already given of Algebra, and to shew, that letters are there used, not so much to signify particular quantities as such, as to signify the relation they have to one another in any problem or computation. From all which it may be observed, that letters represent quantities in Algebra just in the same manner as they do persons in common life, when two or more persons are distinctly to be considered with regard to any compact, law-suit, or in any other relation whatever.

N. B. A single quantity is sometimes represented by two or more letters, when it is considered as the product of the quantities signified by those letters singly: thus ab is the product of the multiplication of a and b ; and abc is the product arising from the continual multiplication of a , b and c . But of this more particularly under the head of multiplication.

Of affirmative and negative quantities in Algebra.

2. Algebraic quantities are of two sorts, affirmative and negative: an affirmative quantity is a quantity greater than nothing, and is known by this sign $+$; a negative quantity is a quantity less than nothing, and is known by this sign $-$: thus $+a$ signifies that the quantity a is affirmative, and is to be read thus, *plus a* , or more a : $-b$ signifies that the quantity b is negative, and must be read thus, *minus b* , or less b .

The possibility of any quantity's being less than nothing is to some a very great paradox, if not a downright absurdity; and truly so it would be, if we should suppose it possible for a body or substance to be less than nothing. But quantities, whereby the different degrees of qualities are estimated, may be easily conceived to pass from affirmation through nothing into negation. Thus a person in his fortunes may be said to be worth 2000 pounds, or 1000, or nothing, or -1000 , or -2000 ; in

in which two last cases he is said to be 1000 or 2000 pounds worse than nothing: thus a body may be said to have 2 degrees of heat, or one degree, or no degree, or — one degree, or — two degrees: thus a body may be said to have two degrees of motion downwards, or one degree, or no degree, or — one degree, or — two degrees, &c. Certain it is, that all contrary quantities do necessarily admit of an intermediate state, which alike partakes of both extremes, and is best represented by a cypher or 0: and if it is proper to say, that the degrees on either side this common limit are greater than nothing; I do not see why it should not be as proper to say of the other side, that the degrees are less than nothing; at least in comparison to the former. That which most perplexes narrow minds in this way of thinking, is, that in common life, most quantities lose their names when they cease to be affirmative, and acquire new ones so soon as they begin to be negative: thus we call negative goods, debts; negative gain, loss; negative heat, cold; negative descent, ascent, &c: and in this sense indeed, it may not be so easy to conceive, how a quantity can be less than nothing, that is, how a quantity under any particular denomination, can be said to be less than nothing, so long as it retains that denomination. But the question is, whether, of two contrary quantities under two different names, one quantity under one name may not be said to be less than nothing, when compared with the other quantity, though under a different name; whether any degree of cold may not be said to be further from any degree of heat, than is lukewarmth, or no heat at all. Difficulties that arise from the imposition of scanty and limited names, upon quantities which in themselves are actually unlimited, ought to be charged upon those names, and not upon the things themselves, as I have formerly observed upon another occasion; see introduction, art. 11. In Algebra, where quantities are abstractedly considered, without any regard to degrees of magnitude, the names of quantities are as extensive as the quantities themselves; so that all quantities that differ only in degree one from another, how contrary soever they may be one to another, pass under the same name; and affirmative and negative quantities are only distinguished by their signs, as was observed before, and not by their names; the same letter representing both: these signs therefore in Algebra carry the same distinction along with them as do particles and adjectives sometimes in common language, as in the words convenient and inconvenient, happy and unhappy, good health and bad health, &c.

These affirmative and negative quantities, as they are contrary to one another in their own natures, so likewise are they in their effects, a consideration which if duly attended to, would remove all difficulties concerning the signs of quantities arising from addition, subtraction, multi-

plication, division, &c: for the result of working by affirmative quantities in all these operations is known; and therefore like operations in negative quantities may be known by the rule of contraries.

Before we proceed any further, it may not be amiss to advertise, that if a quantity has no sign before it, it must always be taken to be affirmative; and that if it has no numeral coefficient before it, unity must always be understood: thus $2a$ signifies $+2a$, and a signifies $1a$ or $+1a$.

By the numeral coefficient of a quantity, I mean the number or fraction by which that quantity is multiplied: thus $2a$ signifies twice a , or a taken twice, and the coefficient is 2 : $\frac{3}{4}a$, or $\frac{3^a}{4}$ signifies $\frac{3}{4}$ of the quantity a , and the coefficient is $\frac{3}{4}$.

N. B. The sign of a negative quantity is never omitted, nor the sign of an affirmative one, except when such an affirmative quantity is considered by itself, or happens to be the first in a series of quantities succeeding one another: thus we do not often mention the quantity $+a$, but the quantity a ; nor the series $+a - b - c + d$, but the series $a - b - c + d$. We shall now consider the several operations of algebraic quantities.

Of the addition of algebraic quantities.

3. This article I shall divide into several paragraphs: as

1st, Whenever two or more quantities of the same denomination, and which have the same sign before them, are to be added together, put down the sum of their numeral coefficients with the common sign before it, and the common denominator after it: thus $+2a$ and $+3a$ added together make $+5a$, for the same reason as 2 dozen and 3 dozen added together make 5 dozen: thus again, $-3ab$, $-4ab$ and $-5ab$ when added together make $-12ab$; for the same reason as several debts added together make a greater debt.

2^d, If two quantities of the same denomination which have different signs before them are to be added together, put down only the difference of their numeral coefficients with the common denominator after it, and the sign of the greater quantity before it: for in this case, the quantities to be added being contrary one to another, the less quantity, on which side soever it lies, will always destroy so much of the other, as is equal to itself. Thus $+5a$ added to $-2a$ makes $+3a$; as if a person owes me 5000 pounds upon one account, to whom I owe 2000 upon another, the balance upon the whole will be 3000 pounds on my side. If it be objected, that this is subtraction, and not addition; I answer, that the addition of $-2a$ will at any time have the same effect as the subtraction

subtraction of $+2a$: but I deny that the addition of $-2a$ is the same, or will have the same effect as the subtraction of $-2a$. Other examples of this case may be these; $+7a$ added to $-7a$ gives 0; $+3a$ added $-12a$ gives $-9a$; $+a$ added to $-5a$ gives $-4a$; $+5a$ added to $-a$ gives $+4a$; $+\frac{1}{3}a$ added to $-\frac{1}{4}a$ gives $+\frac{1}{12}a$, &c.

3d, When many quantities of the same denomination are to be added together, whereof some are affirmative and some negative, reduce them first to two, by adding all the affirmative quantities together, and all the negative ones, and then to one by the last paragraph. Thus $+10a-9a+8a-7a$ when added together make $2a$; for $+10a$ and $+8a$ make $+18a$, and $-9a$ and $-7a$ make $-16a$; and $+18a$ and $-16a$ make $+2a$.

4th, Quantities of different denominations will not incorporate, and therefore cannot otherwise be added together, than by placing them in any order one after another with their proper signs before them, except the first, whose sign, if affirmative, may be omitted. Thus $+2a$ and $-3b$ and $+4c$ and $-5d$, when added together, make $2a-3b+4c-5d$: thus a and b added together make $a+b$; and hence it is, that whenever two quantities are found with this sign $+$ betwixt them, it signifies the sum arising from the addition of those two quantities together: thus if a stands for 7, and b stands for 3, $a+b$ will stand for 10, and so of the rest: but if $-b$ is to be added to a , the sum must be written down thus, $a-b$; for to add $-b$, is the same as to subtract $+b$.

5th, Compound quantities, whose members are all of different denominations, are likewise incapable of being added any other way, than by being placed one after another without altering their signs: thus $3a+4b$ added to $5c-6d$ can only make $3a+4b+5c-6d$. But if the members are not all of different denominations, it may then be convenient to place one compound quantity under another, with like parts under like, as far as it can be done, as in the following examples:

$$\begin{array}{r} a+b \\ a-b \\ \hline 2a \quad *. \dagger \end{array}$$

† For a and a added together make $2a$; and $+b$ and $-b$ added together destroy one another, and so make 0 or *; which character in Algebra is always used to signify a vacant place.

$$\begin{array}{r} 2x-3a+4b-5c+6d-7e \quad * \\ 10x+9a-8b-7c-6d \quad *-5f \\ \hline 12x+6a-4b-12c \quad *-7e-5f. \end{array}$$

Note, That in the addition, subtraction and multiplication of compound algebraic quantities, it matters little which way the work is carried

ed on, whether from right to left, or from left to right, because here are no reserves made for higher places.

Of the subtraction of algebraic quantities.

4. Whenever a simple algebraic quantity is to be subtracted from another quantity, whether simple or compound, first change the sign of the quantity to be subtracted, that is, if it ~~is~~ be affirmative, make it, or at least call it negative, and *vice versa*, and then add it so changed to the other: for since (as was before hinted) the subtracting of any one quantity from another, is the same in effect as adding the contrary, and since changing the sign of the quantity to be subtracted, renders that quantity just contrary to what it was before, it is evident, that after such a change it may be added to the other, and that the result of this addition will be the same with that of the intended subtraction. Thus may the rule of subtraction, by changing the sign of the quantity to be subtracted, be at any time changed into that of addition, just as the rule of division in fractions by inverting the terms of the divisor, was changed into that of multiplication. As for example, $+b$ subtracted from a leaves $a-b$, because $-b$ added to a makes $a-b$; so that $a-b$ may be considered either as the sum of a and $-b$ added together, or as the remainder of $+b$ subtracted from a , or as the difference between a and b , or as the excess of a above b , all which amount to the same thing: as if a signifies 7, and b 3, $a-b$ must stand for 4, and so of the rest.

The rule of subtraction here given is universal, though there will not be always occasion to have recourse to it: for suppose $3a$ is to be subtracted from $7a$, every ones common sense will inform him, that here must remain $4a$, just as threescore subtracted from sevenscore leaves fourscore.

Other examples of algebraic subtraction may be these that follow.

1st, $7a$ subtracted from $5a$ leaves $-2a$, because $-7a$ added to $+5a$ makes $-2a$, by the 2^d paragraph of the last article.

2^d, $9a$ subtracted from 0 leaves $-9a$, because $-9a$ added to 0 makes $-9a$.

3^d, $12a$ subtracted from $-3a$ leaves $-15a$, because $-12a$ added to $-3a$ makes $-15a$, by the first paragraph of the last article.

4th, $-3a$ subtracted from $-8a$ leaves $-5a$, because $+3a$ added to $-8a$ makes $-5a$.

5th,

5th, $-7a$ subtracted from $-3a$ leaves $+4a$, because $+7a$ added to $-3a$ makes $+4a$.

6th, $-6a$ subtracted from 0 leaves $+6a$, because $+6a$ added to 0 makes $+6a$.

7th, $-5a$ subtracted from $+5a$ leaves $+10a$, because $+5a$ added to $+5a$ makes $+10a$.

8th, $-b$ subtracted from a leaves $a+b$, because $+b$ added to a makes $a+b$, by the 4th paragraph of the last article.

9th, -2 subtracted from 7 leaves 9, because $+2$ added to 7 makes 9.

From the first of these examples it appears, that a greater quantity may be taken out of a less, but then the remainder will be negative; just as a gamester that has but 5 guineas about him may loose 7, but then there will remain a debt of 2 guineas upon him. By the last example it appears, that -2 subtracted from 7 leaves 9, that is, that if a negative quantity be subtracted from an affirmative one, the affirmative quantity will be so far from being diminished thereby, that it will be increased; a principle which I fear will be found somewhat hard of digestion, especially by weak constitutions: therefore to strengthen my patient as far as lies in my power, I shall suggest to him the following considerations:

1st, In any subtraction, if the remainder and the less number added together, make the greater, the subtraction is just: but in our case, the remainder 9 added to the less number -2 makes the greater number 7; therefore -2 subtracted from 7 leaves 9.

2dly, In all subtraction whatever, the remainder is the difference betwixt the greater number and the less: but the difference between $+7$ and -2 is 9; therefore -2 subtracted from $+7$ leaves 9.

3dly, 7 is equal to $9-2$ by the second paragraph of the last article; therefore -2 subtracted from 7 will have the same remainder as -2 subtracted from $9-2$: but -2 subtracted from $9-2$ leaves 9; therefore -2 subtracted from 7 leaves 9. In short, the taking away a defect in any case whatever, will amount to the same, as adding something real: as if an estate be incumbered with a mortgage or a rent-charge upon it, whoever takes off the incumbrance, just so much increases the value of the estate.

4thly, The less there is taken from 7, the more will be left: if nothing be taken, there will remain 7; therefore if less than nothing be taken, there ought to remain more than 7.

5thly, If after all that has been said, or perhaps all that can be said in this abstracted way, some scruples still remain, let us apply the principle we have already advanced, and try whether we shall meet with any bet-

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ter success that way. Let it then be required to subtract the compound quantity $a - 2$ from the compound quantity $6a + 7$: in order to this, I place a under $6a$, and -2 under 7 , and then subtract as follows; a from $6a$ and there remains $5a$, -2 from 7 and (if our assertion be true) there remains 9 ; therefore the whole remainder is $5a + 9$. Now I dare appeal to every ones common sense, whether this subtraction be not just: for certain it is, that if a be subtracted from $6a + 7$, the remainder will be $5a + 7$; and if so, then it is as certain, that if $a - 2$ be subtracted, which is less than the former by 2 , the remainder will be greater by 2 , that is, $5a + 9$. But to proceed:

Other examples of the subtraction of compound algebraic quantities may be these.

$$\begin{array}{r} a + b \\ a - b \\ \hline * + 2b \dagger \end{array} \quad \begin{array}{l} \dagger \text{ Thus } 7 - 3, \text{ or } 4, \text{ subtracted from } 7 \\ \quad + 3, \text{ or } 10, \text{ leaves twice } 3, \text{ or } 6. \end{array} \quad \begin{array}{r} * + 12 \\ 3a + 7 \\ \hline - 3a + 5. \end{array}$$

$$\begin{array}{r} \text{From } 12x + 6a - 4b - 12c \quad * - 7e - 5f \\ \text{Take } 2x - 3a + 4b - 5c + 6d - 7e \quad * \\ \hline \text{Remains } 10x + 9a - 8b - 7c - 6d \quad * - 5f \\ \text{Proof } 12x + 6a - 4b - 12c \quad * - 7e - 5f. \end{array}$$

If never a member of the subtrahend be found to be of the same denomination with any member of the number from whence the subtraction is to be made, change the sign of every member of the subtrahend, and then add it to the other. As if $5c - 6d$ is to be subtracted from $3a - 4b$, first change the sign of $5c - 6d$, and make it $-5c + 6d$, and then add it to the other, and you will have $3a - 4b - 5c + 6d$ for a remainder.

Of the multiplication of algebraic quantities.

And first, how to find the sign of the product in multiplication, from those of the multiplicator and multiplicand given.

5. Before we can proceed to the multiplication of algebraic quantities, we are to take notice, that if the signs of the multiplicator and multiplicand be both alike, that is, both affirmative, or both negative, the product will be affirmative, otherwise it will be negative: thus $+4$ multiplied into $+3$, or -4 into -3 produces in either case $+12$: but

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multiplication $\frac{-C}{A}$: for the roots of such an equation were $\frac{B+s}{2A}$ and $\frac{B-s}{2A}$; the sum whereof is $\frac{2B}{2A}$, or $\frac{B}{A}$; and if these two roots be multiplied together, their product will amount to $\frac{BB-ss}{4AA}$; but $ss = BB + 4AC$ as was formerly supposed, art. 103; therefore $ss - BB = 4AC$, and $BB - ss = -4AC$; therefore $\frac{BB-ss}{4AA}$, or the product of the two roots equals $\frac{-4AC}{4AA} = \frac{-C}{A}$.

Therefore if $A=1$, that is, if the equation be $xx = Bx + C$, the sum of the roots will be B , and their product $-C$; that is, as the equation now stands, the sum of the roots will be the coefficient of the unknown quantity on the second side of the equation, and their product, what we call the absolute term, with it's sign changed.

Hence we have an easy way to form a quadratic equation whose roots shall be any two given numbers whatever: as for instance, suppose I would have a quadratic equation whose roots shall be the two numbers 3 and 4; here it is plain that the sum of the two numbers 3 and 4 is 7, and that the product of their multiplication is 12; therefore I form an equation whereof one side is xx , and the other side is $7x - 12$, to wit, $xx = 7x - 12$; and the roots of this equation will be the given numbers 3 and 4, as will appear from the resolution: if I intend the two roots to be 3 and -4 , their sum will be -1 , and the product of their multiplication -12 , and the equation $xx = -x + 12$: if the roots are to be -3 and $+4$, their sum will be $+1$, the product of their multiplication -12 , and the equation $xx = x + 12$: lastly, if the roots are to be -3 and -4 , their sum will be -7 , the product of their multiplication $+12$, and the equation $xx = -7x - 12$. I shall demonstrate one general case according to the resolution given in art. 103, which will be sufficient to shew the way to all the rest: let then the roots proposed be p and q , whose sum is $p+q$, and the product of whose multiplication is pq ; and the equation will be $xx = p+q \times x - pq$; now if this equation be referred to the general one, we shall have $A=1$, $B=p+q$, $C=-pq$, $BB = p^2 + 2pq + q^2$, $4AC = -4pq$, $ss = p^2 - 2pq + q^2$, $s = p - q$, $\frac{B+s}{2A} = \frac{p+q+p-q}{2} = \frac{2p}{2} = p$, $\frac{B-s}{2A} =$

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$\frac{p+q-p+q}{2} = \frac{2q}{2} = q$; therefore the two roots of this equation are p and q . Q. E. D.

I think I ought not to omit here, that if any one has a mind to form a quadratic equation with any two given impossible roots whatever, (if I may be allowed the expression,) it may be done by the foregoing rule, provided that these impossible roots be in such a form as is proper for a quadratic equation: as for example, suppose I would form a quadratic equation with these two impossible roots, to wit, $2 + \sqrt{-3}$ and $2 - \sqrt{-3}$, I put ss for -3 ; for though no possible quantity multiplied into itself can produce a negative, yet an impossible one may, that being the very thing wherein the impossibility consists; making then $ss = -3$, I have $s = \sqrt{-3}$, and so the two roots of the equation will now be $2 + s$, and $2 - s$; the sum of these two roots is 4, and the product of their multiplication $4 - ss$; but if $ss = -3$, $-ss = +3$, and $4 - ss = 4 + 3 = 7$; therefore the equation with these roots will be $xx = 4x - 7$: and this will be further evident by the resolution; for if $xx = 4x - 7$, that is, if $xx - 4x = -7$, we shall have $xx - 4x + 4 = -3$, and $x - 2 = \pm \sqrt{-3}$, and $x = 2 + \sqrt{-3}$, or $2 - \sqrt{-3}$.

How to determine the signs of the possible roots of a quadratic equation without resolving it.

109. If all the terms of a quadratic equation be thrown on one side of the equation, so as to be made equal to nothing; and if the term wherein xx , the square of the unknown quantity is concerned, be made the first, that wherein x , the simple power is concerned, be made the second, and the absolute term, as it is called, be made the third; the number of affirmative and negative roots in such an equation may be found by the following rule, to wit, *As often as the signs are changed in passing through all the terms from the first to the last, of so many affirmative roots will the equation consist; but as often as the signs are the same, so many negative roots will be found in the equation.* This is true in all equations whatever, though at present we shall only demonstrate it in the case of a quadratic equation: but first we shall give the following explication of the rule.

CASE I.

Let the equation be $axx - bx + c = 0$. Here there are two changes in passing through the terms from the first to the last, to wit, from
 $+axx$

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$+axx$ to $-bx$, and from $-bx$ to $+c$; therefore the roots of this equation are both affirmative.

CASE 2.

Let the equation be $axx - bx - c = 0$. Here from $+axx$ to $-bx$ is one change, and from $-bx$ to $-c$ is none; therefore this equation consists of an affirmative and a negative root.

CASE 3.

Let the equation be $axx + bx - c = 0$. Here in passing from $+axx$ to $+bx$, there is no change of sign, but in passing from $+bx$ to $-c$ there is a change; therefore this equation also consists of an affirmative and a negative root.

CASE 4.

Lastly, let the equation be $axx + bx + c = 0$. Here there are no changes, and consequently the roots of this equation are both negative. All these cases I shall demonstrate in the following manner.

CASE 1.

Let the equation be $axx - bx + c = 0$, or $axx = bx - c$. Here the product of the two roots is $\frac{c}{a}$ by the last article, that is, the product of the two roots is an affirmative quantity, and therefore those roots must either be both affirmative or both negative; but they cannot be both negative, because their sum is $\frac{+b}{a}$, by the same article; therefore they must both be affirmative.

CASE 2.

Let the equation be $axx - bx - c = 0$, or $axx = bx + c$. Here the product of the two roots is $-\frac{c}{a}$ and consequently those roots must be of different kinds, one affirmative and the other negative; and because their sum, $+\frac{b}{a}$, is an affirmative quantity, it is an argument that the greater root is affirmative.

CASE 3.

Let the equation be $axx + bx - c = 0$, or $axx = -bx + c$. Here again the product of the two roots is $-\frac{c}{a}$, which argues one root to be

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affirmative and the other negative; and because their sum $\frac{-b}{a}$ is a negative quantity, it is an indication that of these two roots, the greater is the negative one.

C A S E 4.

Lastly, let the equation be $axx + bx + c = 0$, or $axx = -bx - c$. Here the product of the two roots is $+\frac{c}{a}$ an affirmative quantity; therefore the roots are either both affirmative or both negative; but they cannot be both affirmative, because their sum $\frac{-b}{a}$ is negative; therefore they must both be negative.

Impossible roots excluded out of the foregoing rule.

The rule here given for determining the number of affirmative and negative roots relates only to possible roots; for impossible ones cannot be said to belong to any class, either of affirmatives or negatives; nay so capricious are they in this respect, that in one and the same equation, the very same impossible roots shall sometimes appear under one form, and sometimes under the other: as for example, this equation $xx + 3 = 0$ may be filled up two ways without affecting either the equation or its roots; to wit, either thus, $xx - 0x + 3 = 0$, the roots of which equation according to the foregoing rule are both affirmative; or thus, $xx + 0x + 3 = 0$, the roots of which equation, though it be the same with the other, and differs only in form, are both negative: the reason of this absurdity is, that the two roots of the equation $xx + 3 = 0$ are impossible, and occasioned this confusion by putting on one shape in one equation, and another shape in the other: this will further appear from the resolution; for if $xx + 3 = 0$, we have $xx = -3$, and $x = +\sqrt{-3}$, or $-\sqrt{-3}$, which are both impossible quantities. Again, the equation $x^3 - 3 = 0$ may be filled up various ways; as thus, $x^3 - 0x^2 + 0x - 3 = 0$, in which equation, according to the foregoing rule, there are three affirmative roots; or thus, $x^3 - 0x^2 - 0x - 3 = 0$, in which equation there is but one affirmative root and two negative ones: hence an experienced Analyst would immediately conclude (as is really the case) that two of the roots of the equation $x^3 - 3 = 0$ were impossible, and that they stood for affirmative quantities in the former way of putting the equation, and for negative ones in the latter. This will further appear, when we come to treat of cubic equations.

Of

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Of biquadratics, and other equations in the form of quadratics.

110. Thus much for the resolution, nature, and properties of a quadratic equation: I shall only add an example or two more of other equations that sometimes put on the form of quadratics, and have done.

EXAMPLE 12.

Let the equation to be resolved be, $\frac{1600}{xx} + xx = 116$; therefore $1600 + x^4 = 116xx$; therefore $x^4 = 116xx - 1600$. This equation is, properly speaking, a biquadratic, that is, an equation wherein the fourth power of the unknown quantity is concerned: now as every possible quadratic equation has two roots, which will equally answer the condition thereof, so a cubic equation, that is, an equation that rises to the third power of the unknown quantity may have three such roots, a biquadratic four, &c: but the equation $x^4 = 116xx - 1600$, though it be a biquadratic, and admits of four roots, yet it is in the form of a quadratic, if we consider xx as the unknown quantity; in which case x^4 must be looked upon as the square of the unknown quantity, and the equation must be referred to the general one in art. 103, thus; $A=1$, $B=116$, $C=-1600$, $BB=13456$, $4AC=-6400$, $ss=7056$, $s=84$, $\frac{B+s}{2A}=100$, $\frac{B-s}{2A}=16$; therefore in this equation, $xx=100$, or 16 : now if $xx=100$, we shall have $x=+$ or -10 ; if $x^2=16$, we shall have $x=+$ or -4 ; therefore the four roots of this biquadratic equation are, $+10$, -10 , $+4$ and -4 : but though in this equation x has four significations, xx has but two, *viz.* 100 and 16, either of which being substituted instead of xx in the original equation, will answer that equality, as may easily be tried.

N. B. Whenever of the four roots of a biquadratic equation any two are equal and contrary to the other two, the equation will be in form of a quadratic, and may be resolved accordingly.

EXAMPLE 13.

Let the equation be $\frac{576}{xx} - xx = 55$: here we have $576 - x^4 = 55xx$, and $x^4 + 55xx = 576$, and $x^4 = -55x^2 + 576$; therefore according to the general equation in art. 103, $A=1$, $B=-55$, $C=576$, $BB=3025$, $4AC=2304$, $ss=5329$, $s=73$, $\frac{B+s}{2A}=9$, $\frac{B-s}{2A}$

$\frac{B-s}{2A} = -64$; therefore in this equation, $xx = +9$ or -64 : if $xx = +9$, $x = +$ or -3 ; if $xx = -64$, x will be equal to $+\sqrt{-64}$, or $-\sqrt{-64}$, both which values are impossible; so that in this equation x has but two values, $+$ or -3 , the other two being impossible; and xx has two values, to wit, $+9$ and -64 , which are both possible, and which being substituted instead of xx into the original equation, will answer that equality. From this example it is easy to see, that a biquadratic equation may have four roots, and never can have more; yet it may sometimes have fewer, upon the account of some of its roots becoming impossible; nay instances might easily be given wherein all the roots of a biquadratic equation are impossible.

If any one disapproves of the resolutions here given, he may perhaps relish the following better: let the equation be $Ax^4 = Bx^2 + C$; here putting z for xx , and consequently zz for x^4 , the equation will be changed into this common quadratic, $Azz = Bz + C$; which being resolved, z or xx , and consequently x itself will be known: suppose the equation to be $Ax^6 = Bx^3 + C$; here putting z for x^3 , the equation will be changed into a quadratic, as before, to wit, $Azz = Bz + C$, the resolution whereof will give z or x^3 , and consequently x by an extraction of the cube root: lastly, let the equation be $Ax = B\sqrt{x} + C$; here putting zz for x , and z for \sqrt{x} , the equation will be $Azz = Bz + C$, as before; whence z , and consequently zz or x will be known.

The solution of some problems producing quadratic equations.

PROBLEM 69.

III. It is required to divide the number sixty into two such parts, that the product of their multiplication may amount to eight hundred sixtyfour.

SOLUTION.

Put x for one of the parts; then will the other part be $60 - x$, and the product of their multiplication will be $60x - xx$; whence the equation will be $60x - xx = 864$; therefore $xx + 864 = 60x$, and $xx = 60x - 864$: this equation compared with the general one in art. 103, gives $A=1$, $B=60$, $C=-864$, $BB=3600$, $4AC=-3456$, $ss=144$, $s=12$, $\frac{B+s}{2A} = 36$, $\frac{B-s}{2A} = 24$; therefore the parts sought are 24 and 36; which upon trial will answer the conditions of the problem.

Observa-

Observations upon the foregoing problem.

OBSERVATION 1st.

In this problem we may clearly see the necessity of the unknown quantity's having sometimes two distinct values in one and the same equation: for here, if I put x for the greater part of 60, the less will be $60 - x$, and the equation will be $60x - xx = 864$: suppose now I put x for the less part; then the greater will be $60 - x$, and the equation will still be $60x - xx = 864$; therefore, whether x be put for the greater or the less part, we still fall into the same equation $60x - xx = 864$; whence I infer, that this equation must either give us both the parts sought, or neither; since no reason can be shewn why it should give us one part rather than the other.

OBSERVATION 2d.

Hence also we see the necessity sometimes of impossible roots, to wit, when the cases of problems to be solved by them become impossible: as for instance, if any number, as 60, be divided into two parts, the nearer the two parts approach towards an equality, the greater will be the product of their multiplication; and therefore if the parts be equal, the product will be the greatest possible: thus if the parts be 24 and 36, the product will be 864; if they be 25 and 35, the product will be 875; if 30 and 30, the product will be 900, which will be the greatest possible: let us now for once put an impossible case, and let it be required to divide the number 60 into two such parts that the product of their multiplication may amount to 901; here the equation will be $60x - xx = 901$;

which being resolved according to art. 103, gives $x = \frac{60 + \sqrt{-4}}{2}$, or

$\frac{60 - \sqrt{-4}}{2}$; but these values of x may be reduced to more simple

terms thus; $-4 = -1 \times +4$ therefore $\sqrt{-4} = \sqrt{-1} \times \sqrt{+4} = \sqrt{-1} \times 2$; therefore $\frac{\sqrt{-4}}{2} = \sqrt{-1}$; but $\frac{60}{2} = 30$; therefore the

two parts sought are $30 + \sqrt{-1}$, and $30 - \sqrt{-1}$, both which are impossible upon the account of the impossibility of $\sqrt{-1}$; and yet these two parts abstractedly considered will answer the conditions of the problem; for if $\sqrt{-1}$ be made equal to s , the two parts will be $30 + s$ and

and $30 - s$; whose sum is 60, and the product of whose multiplication is $900 - ss$; but if $s = \sqrt{-1}$, we shall have $ss = -1$, and $-ss = +1$, and $900 - ss = 901$; therefore the product of the two parts, $30 + \sqrt{-1}$, and $30 - \sqrt{-1}$ amount to 901, as was required.

OBSERVATION 3d.

Lastly, we here also see the necessity of both the roots of a quadratic equation becoming impossible at once. Two impossible quantities added together, may sometimes make a possible one, because one quantity may be as much impossible one way as the other is the contrary way: thus the two impossible quantities $30 + \sqrt{-1}$ and $30 - \sqrt{-1}$ being added together make 60, the impossible surds $+\sqrt{-1}$ and $-\sqrt{-1}$ destroying one another; but a possible and an impossible quantity when added together can never make a possible one; and therefore the two parts of 60 in this problem must either be both possible, or both impossible.

PROBLEM 70.

112. *There are three numbers in continual proportion, whereof the middle term is sixty, and the sum of the extremes one hundred twentyfive: What are the extremes?*

SOLUTION,

For the extremes put x and $125 - x$, and you will have this proportion; x is to 60 as 60 is to $125 - x$, whence by multiplying extremes and means, you have this equation, $125x - xx = 3600$, or $xx + 3600 = 125x$, or $x^2 = 125x - 3600$: here then $A = 1$, $B = 125$, $C = -3600$, $BB = 15625$, $4AC = -14400$, $ss = 1225$, $s = 35$, $\frac{B+s}{2A} = 80$, $\frac{B-s}{2A} = 45$; therefore in this equation, $x = 45$, or 80; but x represents either extreme, because, which extreme soever x is put for, the other will be $125 - x$, and the same equation will arise, to wit, $125x - xx = 3600$; therefore the two extremes are 45 and 80; and they will answer the conditions of the problem; for 45 is to 60 as $\frac{45}{15}$ is to $\frac{60}{15}$, that is, as 3 to 4; and 60 is to 80 as $\frac{60}{20}$ is to $\frac{80}{20}$, which is also as 3 to 4.

PROBLEM 71.

113. *It is required, having given the sum or the difference of two numbers, together with the sum of their squares, to find the numbers,*

SOLU-

SOLUTION.

Case 1st. Let the sum of the numbers sought be 28, and the sum of their squares 400; then putting x and $28-x$ for the two numbers sought, the square of the former will be xx , the square of the latter $784-56x+xx$, and the sum of their squares $2xx-56x+784=400$; and the same equation will arise, whether x be made to stand for one number or the other; therefore the two values of x in this equation will be the two numbers sought; but if $2xx-56x+784=400$, we shall have $2xx-56x=-384$; divide the whole by 2 for a more simple equation, and you will have $xx-28x=-192$; and $xx=28x-192$; which equation being resolved according to art. 103, gives $x=12$, or 16; therefore 12 and 16 are the two numbers sought.

Case 2d. Let now the difference of two numbers be given, suppose 4, and let the sum of their squares be 400, as before; then putting x for the less number, and $x+4$ for the greater, the sum of their squares will be $2xx+8x+16=400$; whence $2xx+8x=384$, $xx+4x=192$, $xx+4x+4=196$, $x+2=\pm 14$, $x=\pm 12$ or -16 ; now it cannot be supposed that $+12$ and -16 are the two numbers required in the problem, for their difference is 30, not 4; neither ought it to be expected; for when x was put for the less number, and $x+4$ for the greater, the equation was $2xx+8x+16=400$; but if x be put for the greater number, and consequently $x-4$ for the less, the equation will be $2xx-8x+16=400$, different from the former; since then a different equation arises according as x is put for the greater or less number, it cannot be expected that one and the same equation should give both: the true state of the case is this; there are two pairs of numbers which will equally solve this question, and the equation $2xx+8x+16=400$ gives the lesser number of each pair; for if we make $x=12$, and $x+4=16$, the numbers 12 and 16 will solve the problem; on the other hand, if we make $x=-16$, we shall have $x+4=-12$, and the numbers -16 and -12 will equally solve the problem; for their difference is $+4$, and the sum of their squares $+400$: here then we may observe, that affirmative and negative solutions of problems are of equal estimation in the nature of things, though perhaps not amongst men, the narrowness of our minds contracting our views; but truth does justice alike to all: certainly negative numbers differ no more from affirmative ones, than affirmative ones do from one another, which is in degree, not in kind; and therefore in the nature of things, negative quantities ought no more to be excluded out of the scale of number, than affirmative ones, though in common life they are set aside.

PROBLEM 72.

114. *What two numbers are those, whose sum is seventeen, and the sum of their cubes one thousand three hundred forty three?*

SOLUTION.

For the two numbers sought put x and $17-x$, and the cube of the former will be xxx , and the cube of the latter $4913-867x+51xx-xxx$, as appears from the following computation:

$$\begin{array}{r}
 17-x \\
 17-x \\
 \hline
 289-17x+xx \\
 \quad -17x \\
 \hline
 289-34x+xx \\
 17-x \\
 \hline
 4913-578x+17xx-x^3 \\
 \quad -289x+34xx \\
 \hline
 4913-867x+51xx-x^3.
 \end{array}$$

Therefore the sum of these two cubes will be $51xx-867x+4913=1343$, and the equation will be the same, whichever of the two numbers sought x is made to stand for; but if $51xx-867x+4913=1343$, we shall have $51xx-867x=-3570$; divide the whole by 51, which though not necessary, is however convenient, to render the equation more simple, since it may be done without fractions, and you will have, $xx-17x=-70$; which being reduced as in art. 103, gives $x=7$, or 10; therefore 7 and 10 are the two numbers sought.

PROBLEM 73.

115. *Let there be a square whose side is a hundred and ten inches; it is required to assign the length and breadth of a rectangled parallelogram or long square, whose perimeter shall be greater than that of the square by four inches, but whose area shall be less than the area of the square by four square inches.*

N. B. By the perimeter of a plain figure is meant the length of a line that will encompass it round; so that the perimeter of a square is equal to four times it's side; and the perimeier of a rectangled parallelogram is equal to twice it's length and twice it's breadth added together.

SOLUTION.

Since the side of the given square is 110 inches, it's area will be 12100 square inches; therefore the area of the parallelogram sought will

Art. 115, 116. PRODUCING QUADRATIC EQUATIONS. 195

will be 12096 square inches: again, the perimeter of the given square is 440 inches; therefore the perimeter of the parallelogram sought must be 444 inches; therefore half it's perimeter, or it's length and breadth added together must be 222 inches; therefore, if either the length or breadth be called x , the other will be $222 - x$, and the area will be $222x - xx = 12096$; which equation resolved according to art. 103, will give $x = 96$, or 126; therefore the breadth of the parallelogram sought must be 96 inches, and the length 126 inches: and these numbers will answer the conditions of the question; for twice the length will be 252, twice the breadth 192, and the whole perimeter 444; moreover 126×96 , or the area will be 12096, as the problem requires.

SCHOLIUM.

This problem shews how grossly they are mistaken who think to estimate the areas or magnitudes of plain figures by their perimeters, as if such figures were greater or less in proportion as their perimeters were so; whereas here we see, that the perimeter of one figure may be greater than that of another by four inches, and at the same time it's area may be less than the area of that other by four square inches. This error, it is true, does not obtain but in low and vulgar minds, nor there neither any longer than whilst it continues to be a matter of mere speculation, and truth and falsehood are equally indifferent to them: for whenever men come to apply their notions, and find it their interest not to be mistaken, then it is, and frequently not till then, that they begin to look about them, correct their errors, and entertain more just and accurate notions of things. The greatest part of mankind have a natural aversion to abstract thinking, and where their interest is not concerned, will rather submit their opinions to humour, caprice and custom, or be content to be without any opinions at all, than they will examine strictly into the nature of things.

PROBLEM 74.

116. *One buys a certain number of Oxen for eighty guineas; where it must be observed, that if he had bought four more for the same money, they would have come to him a guinea apiece cheaper: What was the number of oxen?*

SOLUTION.

For the number of oxen put x ; then to find the price of a single ox, say, if x oxen cost 80 guineas, what will one ox cost? and the answer is $\frac{80}{x}$; and for the same reason, if he had bought 4 more, that is, $x+4$

B b 2

for

for the same money, the price of an ox would have been $\frac{80}{x+4}$; but according to the problem, the latter price is less than the former by one guinea; whence we have this equation, $\frac{80}{x} - 1 = \frac{80}{x+4}$; therefore 80

$-x = \frac{80x}{x+4}$; therefore $80 - x \times 4 + x$ or $320 + 76x - xx = 80x$ therefore $xx + 80x = 76x + 320$; therefore $xx = -4x + 320$. Here then $A=1$, $B=-4$, $C=320$, $BB=16$, $4AC=1280$, $ss=1296$, $s=36$, $\frac{B+s}{2A}=16$, $\frac{B-s}{2A}=-20$; therefore $x=+16$, or -20 ; therefore the number of oxen was 16, the negative root -20 having no place in this problem; and this number 16 answers the condition of the problem; for if 16 oxen cost 80 guineas, one will cost 5 guineas; but if 20 oxen cost 80 guineas, one will cost 4 guineas.

N. B. The equation $\frac{80}{x} - 1 = \frac{80}{x+4}$, gave $x=+16$ or -20 , not because the number -20 would solve the problem, but because it would solve the equation; for if we make $x=-20$, we shall have $\frac{80}{x} = -4$, and $\frac{80}{x} - 1 = -5$; on the other side we shall have $x+4=-16$, and $\frac{80}{x+4} = -5$; therefore if x be made equal to -20 , we shall have $\frac{80}{x} - 1 = \frac{80}{x+4}$, because both sides are equal to -5 ; and so in all other cases we shall always find, that the several roots of an equation will be such as will equally solve that equation, though perhaps they may not be equally proper to solve the problem from whence the equation was deduced: but of this more in another place.

PROBLEM 75.

117. *A certain company at a tavern had a reckoning of seven pounds four shillings to pay; upon which two of the company sneaking off, obliged the rest to pay one shilling apiece more than they should have done: What was the number of persons?*

SOLUTION.

For the number of persons put x ; then to find the number of shillings every man should have paid, say, if x persons were to have paid 144 shillings,

Art. 117, 118, 119. PRODUCING QUADRATIC EQUATIONS. 197

shillings, what must one man have paid? and the answer is $\frac{144}{x}$; therefore $\frac{144}{x}$ is the number of shillings every man should have paid; and for the same reason $\frac{144}{x-2}$ is the number of shillings every man did pay; but according to the problem, this latter reckoning is greater than the former by one shilling; whence the equation will be $\frac{144}{x} + 1 = \frac{144}{x-2}$; therefore $144 + x = \frac{144x}{x-2}$; therefore $x-2 \times 144 + x$, or $xx + 142x - 288 = 144x$; therefore $xx - 288 = 2x$; therefore $xx = 2x + 288$. Here then $A=1$, $B=2$, $C=288$, $BB=4$, $4AC=1152$, $ss=1156$, $s=34$, $\frac{B+s}{2A}=18$, $\frac{B-s}{2A}=-16$; therefore $x=+18$, or -16 ; but negative roots have no place in this sort of problems; therefore the number of persons was 18, which answers the condition; for $\frac{144}{18}=8$, and $\frac{144}{16}=9$.

PROBLEM 76.

118. *What number is that, which being added to it's square root will make two hundred and ten?*

SOLUTION.

For the number sought put xx ; then will it's square root be x , and the equation will be $xx + x = 210$, or $xx = -x + 210$; where $A=1$, $B=-1$, $C=210$, $BB=1$, $4AC=840$, $ss=841$, $s=29$, $\frac{B+s}{2A}=14$, $\frac{B-s}{2A}=-15$; therefore $x=+14$, or -15 ; therefore xx or the number sought, equals 196 or 225, supposing the square root of 225 to be -15 ; and either of these two numbers will answer the condition; for $196 + 14 = 210$, and $225 - 15 = 210$.

PROBLEM 77.

119. *What two numbers are those, the product of whose multiplication is one hundred ninety two, and the sum of whose squares is six hundred and forty?*

SOLU-

SOLUTION.

For the two numbers sought put x and $\frac{192}{x}$; then will the square of the former be xx , and that of the latter $\frac{36864}{xx}$, and the sum of their squares will be $xx + \frac{36864}{xx} = 640$; which equation will be the same, whichever of the two numbers sought x is made to stand for; but if $xx + \frac{36864}{xx} = 640$, we shall have $x^4 + 36864 = 640xx$; and $x^4 = 640x^2 - 36864$: here then $A=1$, $B=640$, $C=-36864$, $BB=409600$, $4AC=-147456$, $ss=262144$, $s=512$, $\frac{B+s}{2A}=576$, $\frac{B-s}{2A}=64$; therefore $xx=576$, or 64 ; therefore $x=+$ or -24 , or $+$ or -8 ; therefore the two numbers sought are 8 and 24.

PROBLEM 78.

120. One lays out a certain sum of money in goods, which he sold again for twenty four pounds, and gained as much per cent as the goods cost him: I demand what they cost him.

N. B. One's gain per cent is so much as he gains, every hundred pounds he lays out; or if he does not lay out so much as a hundred pounds, his gain per cent however, is so much as he would have gained if he had laid out a hundred pounds with the same advantage: thus if he lays out 20 pounds and gains 2 pounds, he is said to make 10 per cent of his money, because 20 pounds is to 2 pounds as 100 pounds is to 10 pounds.

SOLUTION.

Put x for the money laid out, and the gain will be $24-x$; say then by the golden rule, if in laying out x he gained $24-x$, what would he have gained if he had laid out 100 pounds to the same advantage? and the answer will be $\frac{2400-100x}{x}$; therefore $\frac{2400-100x}{x}$ will be his gain per cent; but according to the problem, this gain is equal to x , the money laid out; therefore $x = \frac{2400-100x}{x}$, and $xx = 2400 - 100x$: here then $A=1$, $B=-100$, $C=2400$, $BB=10000$, $4AC=9600$,

$= 9600$, $ss = 19600$, $s = 140$, $\frac{B+s}{2A} = 20$, $\frac{B-s}{2A} = -120$; therefore the money laid out was 20 pounds; therefore his gain *per* 20 was 4 pounds; therefore his gain *per cent* was 20 pounds, equal to the money laid out.

PROBLEM 79.

121. One lays out thirty three pounds fifteen shillings in cloth, which he sold again for forty eight shillings per piece, and gained as much in the whole as a single piece cost: I demand how he bought in his cloth per piece.

SOLUTION.

Put x for the number of shillings every single piece was bought for, and the gain *per* piece will be $48 - x$; say then by the rule of proportion, if in laying out x he gained $48 - x$, what did he gain in laying out 33 pounds 15 shillings, or 675 shillings? and the answer will be $\frac{32400 - 675x}{x}$; therefore $\frac{32400 - 675x}{x}$ will be his whole gain; but according to the problem, the whole gain was equal to x , the money given for a single piece; therefore $x = \frac{32400 - 675x}{x}$ therefore $xx = 32400 - 675x$; therefore $A = 1$, $B = -675$, $C = 32400$, $BB = 455625$, $4AC = 129600$, $ss = 585225$, $s = 765$, $\frac{B+s}{2A} = 45$, $\frac{B-s}{2A} = -720$; therefore $x = +45$, or -720 ; therefore the money every single piece was bought for, was 45 shillings, and the gain *per* piece was 3 shillings; but if 45 shillings gains 3 shillings, 33 pounds 15 shillings, or 675 shillings, will gain 45 shillings; therefore the whole gain was 45 shillings, equal to the money given for a single piece.

N. B. It is not impossible but that sometimes two different problems may produce one and the same equation; and then the equation must provide equally for both: therefore in such a case, though the equation has two roots, and both affirmative, yet it must not be expected that both roots should equally serve for the solution of one problem, and that there should be no solution left for the other; we ought rather to conclude, whenever an equation gives two roots, and both affirmative, whereof one only will solve the problem that produced the equation, we ought, I say, rather to conclude, that the other root is for the solution of some other problem producing the same equation; a curious instance whereof we have in the two following problems.

PRO-

PROBLEM 80.

122. Two travellers A and B, set out from two places C and D at the same time, A from C bound for D, and B from D bound for C; when they met and had computed their travels, it was found, that A had travelled thirty miles more than B, and that at their rate of travelling, A expected to reach D in four days, and B to reach C in nine days: I demand the distance between the two places C and D.

SOLUTION.

Put x for the number of miles between C and D; then it is plain that A and B both together had travelled x miles when they met; therefore as much as the miles travelled by A exceeded $\frac{x}{2}$, just so much did the miles travelled by B come short of $\frac{x}{2}$; but by the supposition, A's miles exceeded those of B by 30; therefore A must have travelled $\frac{x}{2} + 15$ or $\frac{x+30}{2}$ miles; and B must have travelled $\frac{x}{2} - 15$ or $\frac{x-30}{2}$ miles; therefore the remaining part of A's journey is $\frac{x-30}{2}$ miles, which he expects to perform in four days, and the remaining part of B's journey is $\frac{x+30}{2}$ miles, which he expects to perform in 9 days: these things being allowed, let us now enquire into the number of days each hath travelled already; and first for A say, if A expects to travel $\frac{x-30}{2}$ miles in 4 days, in how many days did he travel $\frac{x+30}{2}$ miles? and the

answer is $\frac{4 \times \frac{x+30}{2}}{\frac{x-30}{2}} = \frac{4 \times x + 30}{x-30}$; then for B say, if B expects to

travel $\frac{x+30}{2}$ miles in 9 days, in how many days did he travel $\frac{x-30}{2}$ miles? and the answer is $\frac{9 \times x - 30}{x+30}$; therefore A hath travelled $\frac{4 \times x + 30}{x-30}$ days,

days, and $B \frac{9x-30}{x+30}$ days from the time of their first setting out; but as they both set out at the same time, and are now met, they must both have travelled the same number of days; therefore $\frac{4x+30}{x-30} = \frac{9x-30}{x+30}$: multiply both sides of the equation into $x-30$, and you will have $4x+30 = \frac{9x-30 \times x-30}{x+30}$; again, multiply by $x+30$, and you will have $4x+30 \times x+30 = 9x-30 \times x-30$; extract the square root of both sides, and you will have $\pm 2x+30 = \pm 3x-30$: this general equation resolves itself into 4 particular ones, *viz.*

$$1^{\text{st}}, \quad +2x+30 = +3x-30.$$

$$2^{\text{d}}, \quad +2x+30 = -3x-30.$$

$$3^{\text{d}}, \quad -2x+30 = +3x-30.$$

$$4^{\text{th}}, \quad -2x+30 = -3x-30.$$

But as the two last of these equations give but the same values as the two former, I shall only make use of the two former, thus;

1st, Suppose $+2x+30 = +3x-30$, then we shall have $2x+60 = 3x-90$, and $x = 150$.

2^{dly}, Suppose $+2x+30 = -3x-30$, then we shall have $2x+60 = -3x+90$, and $x = 6$; therefore the distance between the two places *C* and *D* must either be 150 miles, or 6 miles; but 6 miles it cannot be, because when *A* came up to *B*, he had travelled 30 miles more than *B*, and had not yet reached *D*; therefore the distance between the two places *C* and *D* must be 150 miles; which will satisfy the problem; for then *A* must have travelled $75+15$, or 90 miles, and *B* $75-15$ or 60 miles, from the time of their setting out; therefore *A* has 60 miles, and *B* 90 to travel; but if *A* could travel 60 miles in 4 days, he must, at the same rate, have travelled 90 miles in 6 days, and if *B* could travel 90 miles in 9 days, he must have travelled 60 miles also in 6 days; therefore they both travelled the same number of days from the time of their first setting out to the time of their meeting, as the problem requires.

PROBLEM 81.

123. Two travellers *A* and *B*, set out from two places *C* and *D* at the same time; *A* from *C* with a design to pass through *D*, and *B* from *D* with a design to travel the same way: after *A* had overtaken *B*, and they had computed their travels, it was found, that they had both together travelled thirty miles, that *A* had passed through *D* four days before, and that *B* at his rate of travelling, was a nine days journey distant from *C*: I demand the distance between the two places *C* and *D*.

SOLUTION.

Put x for the number of miles from *C* to *D*; then it is plain, that *A* must have travelled more miles than *B* by x ; but they both together travelled 30 miles, by the supposition; therefore as much as *A*'s miles exceeded 15, just so much must *B*'s miles come short of 15; but the whole difference was x , as above; therefore *A* must have travelled $15 + \frac{x}{2}$, or $\frac{30+x}{2}$ miles, and *B* must have travelled $15 - \frac{x}{2}$, or $\frac{30-x}{2}$ miles; therefore *A*'s distance from *D*, after he had overtaken *B*, was $\frac{30-x}{2}$ miles, which he had travelled in 4 days, and *B*'s distance from *C* was $\frac{30+x}{2}$ miles, which by the problem he could travel in 9 days; therefore to find how many days each had travelled already, say, if *A* hath travelled $\frac{30-x}{2}$ miles from *D* in 4 days, in how many days did he travel $\frac{30+x}{2}$ miles since his departure from *C*? and the answer is

$$4 \times \frac{30+x}{2} = \frac{4 \times 30+x}{\frac{30-x}{2}}; \text{ again say, if } B \text{ could travel } \frac{30+x}{2} \text{ miles, the}$$

whole distance from *C*, in 9 days, in how many days did he travel $\frac{30-x}{2}$

miles since his setting out from *D*? and the answer is $\frac{9 \times 30-x}{30+x}$; but as they both set out at the same time, and *A* has now overtaken *B*, they must both have travelled the same number of days; therefore we have
this

this equation, $\frac{4 \times 30 + x}{30 - x} = \frac{9 \times 30 - x}{30 + x}$: multiply both sides into $30 - x$,

and you will have $4 \times 30 + x = \frac{9 \times 30 - x \times 30 - x}{30 + x}$; again, multiply

by $30 + x$, and you will have $4 \times 30 + x \times 30 + x = 9 \times 30 - x \times 30 - x$;

but the product of $30 - x \times 30 - x$ differs nothing from the product of $x - 30 \times x - 30$, as will appear upon trial, and will be further evi-

dent from hence, that $30 - x$ and $x - 30$ differ no more from one another than an affirmative quantity does from an equal negative one, and therefore each multiplied into itself must give the same product; therefore the

equation as it now stands is, $4 \times x + 30 \times x + 30 = 9 \times x - 30 \times x - 30$; but this equation is the same with the equation deduced from the last problem, which justifies what I observed before, art. 121, that different problems may produce the same equation; therefore the two roots of this equation will be 6 and 150, as in the last article; therefore the distance between the two places *C* and *D* must either be 6 miles, or 150 miles; but 150 miles it cannot be, because, after *A* had passed from *C* beyond *D*, and at last had overtaken *B*, they had both travelled but 30 miles; therefore the distance from *C* to *D* must be 6 miles; and this number will answer the conditions of the problem; for then *A*, when he had overtaken *B*, had travelled $15 + 3$ or 18 miles, and *B* $15 - 3$ or 12 miles; therefore *A* had got 12 miles beyond *D* in 4 days time, and *B* was 18 miles distant from *C*, which he could travel in 9 days; but at the rate of 12 miles in 4 days, *A* must have performed his 18 miles journey in 6 days; and at the rate of 18 miles in 9 days, *B* must have performed his 12 miles journey also in 6 days; therefore from the time of their first setting out to the time of *A*'s overtaking *B*, they had both travelled the same number of days, as the problem requires; therefore the supposition whereupon this calculation was founded, to wit, that the distance of *C* from *D* was 6 miles, is just.

N. B. The solutions here given of the two last problems, are in my opinion, the most natural, though somewhat different from the rest.

A L E M M A.

124. *The sum of a series of quantities in arithmetical progression may be had, by adding the greatest and least terms together, and then multiplying either half that sum by the whole number of terms, or the whole sum by half the number of terms, or lastly, by multiplying the whole sum into the whole number of terms and then taking half the product: thus in the series 2, 4,*

C c 2

6, 8,

6, 8, 10, 12, where the least term is 2, the greatest 12, their sum 14, and the number of terms 6; the sum of all the terms taken together will be 7×6 , or 14×3 , or $\frac{14 \times 6}{2} = 42$. This will best appear by writing down the series 2, 4, 6, 8, 10, 12, and then by writing down over it the same series inverted, 12, 10, 8, 6, 4, 2: for if this be done, 2, the first term of the lower series added to 12, the first term of the upper series, (which is the same as the greatest and least terms of the same series added together) will make 14; in like manner every term of the lower series added to the next above it will make 14; therefore both the serieses together will be equal to 14 as often taken as there are terms in either series, that is, 6 times 14, or 84; therefore either series taken alone will be equal to 42.

$$\begin{array}{rcccccc}
 12 & 10 & 8 & 6 & 4 & 2 \\
 2. & 4. & 6. & 8. & 10. & 12. \\
 \hline
 14. & 14. & 14. & 14. & 14. & 14.
 \end{array}$$

The design of this lemma is, to add the terms of a series together, where only the greatest and least terms and the number of terms are known, or supposed to be known; the intermediate terms being either not assigned, or too many to be summed up by a continual addition.

PROBLEM 82.

125. *A traveller, as A, sets out from a certain place, and travels one mile the first day, two miles the second day, three the third, four the fourth, &c; and five days after, another, as B, sets out from the same place, and travels the same road at the rate of twelve miles every day: I demand how long and how far A must travel before he is overtaken by B.*

SOLUTION.

Put x for the number of days *A* travelled before he was overtaken by *B*; then to find an expression for the number of miles travelled by him in that time, I observe that in three days *A* travelled over $1 + 2 + 3$ miles, that is, he travels over a series of miles in arithmetical progression, whereof the number of terms is 3, the greatest term 3, and the least term 1; in four days he travels over a series whereof the number of terms is 4, the greatest term 4, and the least 1; therefore universally, in any number x of days, he must travel over a series of miles in arithmetical progression, whereof the number of terms is x , the greatest term x , and the least term 1; but the sum of the extremes of this series is $x + 1$, which multiplied by x the number of terms, gives $xx + x$, the half where-

whereof is $\frac{xx+x}{2}$; therefore by the lemma foregoing, $\frac{xx+x}{2}$ will be the sum of this series, and consequently the miles travelled by A before he was overtaken: again, if A travels x days, B must have travelled $x-5$ days, which at the rate of 12 miles a day, gives $12x-60$ for the miles travelled by B when he overtook A ; but as they both set out from the same place, and are now got together, they must have travelled the same number of miles; whence we have this equation, $\frac{xx+x}{2} = 12x-60$; therefore $xx+x=24x-120$; therefore $xx=23x-120$; compare this equation with the general one in art. 103, and you will have $A=1$, $B=23$, $C=-120$, $BB=529$, $4AC=-480$, $ss=49$, $s=7$, $\frac{B+s}{2A}=15$, $\frac{B-s}{2A}=8$; therefore $x=8$, or 15: now for the better application of these roots to the solution of this problem, it must be observed, that the problem is more limited than the equation deduced from it; just as if, in translating out of one language into another, the terms of the latter, instead of being adequate to those of the former, should be found to be of a more extensive signification: in the problem it is only supposed that B overtakes A , whereas in the equation it is supposed that A and B are got both together by having travelled the same number of miles from their first setting out, without specifying whether this arises from B 's overtaking A , or from A 's overtaking B ; both which in this case must necessarily happen in the course of their travels, provided they be but continued long enough for that purpose: for since at first, B is the swifter traveller, whenever they come together, it must arise from B 's overtaking A , which happens after A has travelled 8 days; then if we suppose them still to continue their travels, B passes by A , and continues before him for some time; but after 12 days, A becomes the swifter traveller, and must necessarily come up to B again after he has travelled 15 days: therefore though the two roots, 8 and 15, will both answer the condition of the equation, yet but one of them, to wit, 8, will answer the condition of the problem; and that both of them will answer the condition of the equation, will be evident as follows.

In 8 days A travels over a series of miles whereof the number of terms is 8, the greatest 8, and the least 1; the sum of which series is 36 miles; but when A has travelled 8 days, B must have travelled 3 days, during which time, at the rate of 12 miles a day, he also must have travelled 36 miles; therefore after A had travelled 8 days, A and B must necessarily find themselves together: again, in 15 days, A must have travel-

travelled over a series of miles, whereof the number of terms is 15, the greatest 15, the least 1, and the sum 120 miles; but when *A* had travelled 15 days, *B* must have travelled 10 days, which at 12 miles a day gives also 120 miles; therefore now again *A* and *B* must find themselves together; and consequently 8 and 15 equally answer the supposition contained in the equation.

N. B. If we suppose *B* after 5 days to have begun to follow *A*, and to have travelled only 10 miles a day, he could never have overtaken *A*, nor *A* him, so that in this case, both the roots would have become impossible, as will be found by the resolution of an equation founded upon this supposition.

PROBLEM 83.

126. *It is required to divide the number ten into two such parts, that the product of their multiplication being added to the sum of their squares, may make seventy-six.*

SOLUTION.

The two parts sought, x and $10 - x$.

The product of their multiplication, $10x - xx$.

The sum of their squares, $2xx - 20x + 100$.

The product of their multiplication }
added to the sum of their squares, $\left\{ \begin{array}{l} x^2 - 10x + 100 = 76. \end{array} \right.$

Whence $x = 4$, or 6; but this equation will be the same, which part soever x is put for; therefore the two parts sought are 4 and 6.

PROBLEM 84.

127. *It is required to find two numbers with the following properties, to wit, that twice the first with three times the second may make sixty, and moreover, that twice the square of the first with three times the square of the second may make eight hundred and forty.*

SOLUTION.

For the two numbers sought put x and y , and we shall have

Equ. 1st, $2x + 3y = 60$, and

Equ. 2d, $2x^2 + 3y^2 = 840$.

From the first equation, $2x + 3y = 60$, we have

Equ. 3d, $x = \frac{60 - 3y}{2}$; and by squaring both sides

we have Equ. 4th, $xx = \frac{3600 - 360y + 9yy}{4}$.

From

Art. 127, 128. PRODUCING QUADRATIC EQUATIONS.

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From the second equation, $2xx + 3yy = 840$, we have

$$\text{Equ. 5th, } xx = \frac{840 - 3yy}{2}.$$

Compare the two values of xx in the fourth and fifth equations, which must necessarily be equal one to the other, and you will have $\frac{3600 - 360y + 9yy}{4}$

$$= \frac{840 - 3yy}{2}; \text{ multiply both sides into 2, by halving the denominators,}$$

$$\text{and you will have } \frac{3600 - 360y + 9yy}{2} = 840 - 3yy; \text{ therefore } 3600 - 360y$$

$$+ 9yy = 1680 - 6yy; \text{ therefore } 3600 - 360y + 15yy = 1680; \text{ therefore}$$

$$15yy - 360y = -1920; \text{ therefore } 15y = 360y - 1920; \text{ divide by 15}$$

$$\text{for a more simple equation, and you will have } yy = 24y - 128; \text{ whence}$$

$$y = 8, \text{ or } 16: \text{ suppose } y = 8, \text{ then since by the third equation } x =$$

$$\frac{60 - 3y}{2}, \text{ we shall have } x = 18; \text{ suppose } y = 16, \text{ then we shall have } x \text{ or}$$

$$\frac{60 - 3y}{2} = 6; \text{ therefore there are two pair of numbers that will equally}$$

answer the conditions of this problem, to wit, 18 and 8, and also 6 and

16: for a proof, let us first suppose the numbers to be 18 and 8; and

we shall have twice the first number with three times the second $= 36$

$+ 24 = 60$; and twice the square of the first together with three times

the square of the second equal to $648 + 192 = 840$: secondly, let us

suppose the numbers to be 6 and 16; and we shall have twice the first

with three times the second equal to $12 + 48 = 60$; and twice the

square of the first with three times the square of the second equal to

$72 + 768 = 840$.

PROBLEM 85.

128. To find four numbers in continual proportion, and such, that the sum of the two middle terms may be eighteen, and that of the extremes twentyseven.

Note, Four numbers are said to be in continual proportion, when the first is to the second as the second is to the third, and the second is to the third as the third is to the fourth.

SOLUTION.

For the two middle terms put x and y , without intending which is to be the greater; then the extreme next to x may be found by saying,

as y is to x so is x to $\frac{xx}{y}$, and the extreme next to y may be found by

saying,

saying, as x is to y so is y to $\frac{yy}{x}$; therefore the extremes are $\frac{xx}{y}$ and $\frac{yy}{x}$, and their sum $\frac{x^3+y^3}{xy}$; therefore the fundamental equations are 1st, $x+y=18$, or $x=18-y$; and 2dly, $\frac{x^3+y^3}{xy}=27$, or $x^3+y^3=27xy$; instead of x in this equation put $18-y$, it's value in the last, and you will have $x^3=5832-972y+54y^2-y^3$; therefore $x^3+y^3=5832-972y+54yy$; you will also have $27xy$ or $27y \times 18-y=486y-27yy$; therefore $5832-972y+54yy=486y-27yy$; transpose $486y-27yy$, and you will have $81yy-1458y+5832=0$; divide all by 81, which may be done without a fraction, and you will have $yy-18y+72=0$; which equation being resolved, either by the general theorem or any other way, gives $y=6$, or 12 ; and since the equation will be the same, whichever of the two middle terms y stands for, it follows, that the two middle terms are 6 and 12; whence the extreme next to 6 is 3, and that next to 12 is 24; and the numbers are either 3, 6, 12 and 24, or 24, 12, 6 and 3, for either way they will answer the conditions of the problem.

PROBLEM 86.

129. *There are three numbers in continual proportion, whose sum is nineteen, and the sum of their squares one hundred thirtythree: What are the numbers?*

SOLUTION.

For the three numbers sought put x, y and z ; then since by the first condition, x is to y as y is to z , by multiplying extremes and means we have $yy=xz$; again, by the second condition of the problem we have $x+y+z=19$, and $19-y=x+z$, and (squaring both sides) $361-38y+yy=xx+2xz+zz$; subtract yy from one side of the equation, and it's equal xz from the other, and you will have $361-38y=x^2+xz+z^2=x^2+y^2+z^2=133$ by the third condition of the problem: having thus expunged both x and z at once, resolve the equation $361-38y=133$, and you will have y the middle term equal to 6, and $19-y$, or the sum of the extremes $=13$; therefore the problem proposed is now reduced to this, *viz. Of three numbers in continual proportion, whereof fix the middle term, and thirteen the sum of the extremes are given, to find the extremes*: this problem is of the same nature with that in art. 112, and being resolved, gives 4 and 9 for the extremes; therefore the three numbers sought are 4, 6 and 9, or 9, 6 and 4.

P R O-

PROBLEM 87.

130. To find two numbers such, that their difference multiplied into the difference of their squares shall make thirtytwo, but their sum multiplied into the sum of their squares shall make two hundred seventytwo.

SOLUTION.

For the two numbers sought put x and y ; and the first fundamental equation will be $x - y \times x^2 - y^2$, or $x - y \times x - y \times x + y$, or $x^2 - 2xy + y^2 \times x + y = 32$; therefore

$$\text{Equ. 1st, } x^2 - 2xy + y^2 = \frac{32}{x+y}.$$

The second fundamental equation is, $x + y \times x^2 + y^2 = 272$; therefore

$$\text{Equ. 2d, } x^2 + y^2 = \frac{272}{x+y}.$$

From twice the second equation
subtract the first, that is, from

$$2x^2 + 2y^2 = \frac{544}{x+y}$$

subtract

$$x^2 - 2xy + y^2 = \frac{32}{x+y}$$

and you will have

$$x^2 + 2xy + y^2 = \frac{512}{x+y},$$

that is, $\frac{512}{x+y} = \frac{512}{x+y}$; therefore $x+y = 8$, and $x+y = \sqrt[3]{512}$, or the cube root of $512 = 8$: thus we have got the sum of the two numbers sought, to wit, 8; whence their difference may be found by

the first equation, thus; $x^2 - 2xy + y^2 = \frac{32}{x+y}$, that is, $x - y = \frac{32}{8} = 4$; therefore $x - y$, or the difference of the two numbers sought,

equals 2; therefore the problem proposed is now reduced to this; *Having given eight the sum, and two the difference of the two numbers x and y , to find those numbers*; and by art. 26 we shall have $x = 5$, and $y = 3$; which numbers will answer the conditions of the question.

N. B. After we had found $x + y$, the sum of the numbers equal to 8, we might have found the sum of their squares by the second equation, which gave $x^2 + y^2 = \frac{272}{x+y} = \frac{272}{8} = 34$; and then the problem would have been reduced to this; *What two numbers are those, whose sum is eight, and the sum of their squares thirtyfour?* which would have produced a quadratic equation, as in art. 113, whose two roots would have been 5 and 3, as before.

PROBLEM 88.

131. *To find two numbers such, that their difference added to the difference of their squares may make fourteen, and their sum added to the sum of their squares may make twenty six.*

SOLUTION.

For the two numbers sought put x and y , and you will have the two following equations;

$$\text{Equ. 1st, } x - y + x^2 - y^2 = 14.$$

$$\text{Equ. 2d, } x + y + x^2 + y^2 = 26.$$

Add these two equations together, and you will have $2xx + 2x = 40$, $xx + x = 20$, and $x = +4$, or -5 ; again, subtract the first equation from the second, and you will have $2yy + 2y = 12$, $yy + y = 6$, and $y = +2$, or -3 ; and as these two values of y were obtained without any manner of dependence upon those of x , it is plain that either of the values of x may be joined with either of the values of y ; and so we have no fewer than four pairs of numbers which will equally satisfy the conditions of the equations, to wit, $+4$ and $+2$, $+4$ and -3 , -5 and $+2$, -5 and -3 ; but it is the first pair only, which, consisting of affirmative numbers, is proper for the solution of the problem, thus; the difference of 4 and 2 is 2, the difference of their squares 12, and $2 + 12 = 14$; again, the sum of 4 and 2 is 6, the sum of their squares 20, and $6 + 20 = 26$: let us see however how the other pairs will satisfy the conditions of the equations; make then x equal to 4, y , that is, $+y = -3$, and you will have $-y = +3$; whence $x - y = 4 + 3 = 7$, $x^2 - y^2 = 16 - 9 = 7$, and $7 + 7 = 14$; again, $x + y = 4 - 3 = 1$, and $x^2 + y^2 = 16 + 9 = 25$, and $1 + 25 = 26$: in the next place, make $x = -5$, and $y = +2$, then we shall have $x - y = -5 - 2 = -7$, $x^2 - y^2 = 25 - 4 = 21$, and $-7 + 21 = 14$; again, $x + y = -5 + 2 = -3$, and $x^2 + y^2 = 25 + 4 = 29$, and $-3 + 29 = 26$: lastly, make $x = -5$, and $y = -3$, and you will have $x - y = -5 + 3 = -2$, and $x^2 - y^2 = 25 - 9 = 16$, and $-2 + 16 = 14$; again, $x + y = -5 - 3 = -8$, and $x^2 + y^2 = 25 + 9 = 34$, and $-8 + 34 = 26$.

PROBLEM 89.

132. *What two numbers are those, whose sum, when added together, is equal to their product when multiplied together; and this sum or product, when added to the sum of their squares, makes twelve?*

For

SOLUTION.

For the two numbers sought put x and y , and the fundamental equations will be 1st, $x + y = xy$; and secondly, $x + y + x^2 + y^2 = 12$: in the first of these fundamental equations, where $x + y = yx$, we have $yx - x = y$; but $yx - x$ is the product of $y - 1 \times x$, or of $x \times y - 1$;

therefore $x \times y - 1 = y$, and $x = \frac{y}{y-1}$; but if instead of x , this value be substituted into the second fundamental equation, the equation will rise to a biquadratic, for the resolution whereof, no rules have hitherto been given; therefore to extricate ourselves out of this difficulty, it will be proper to have recourse to some other artifice, by trying other positions, as thus; for the sum of the two numbers sought put z ; then will z be also the product of their multiplication, by the supposition; and since this product z added to the sum of their squares gives 12, the sum of their squares will be $12 - z$; but every one knows, that if to the sum of the squares of any two numbers be added their double product, there will arise the square of their sum; therefore $12 - z + 2z$, or $12 + z = z^2$; which equation being resolved, gives $z = +4$, &c; and therefore the question is now reduced to this; *What two numbers are those, whose sum is four, and the product of whose multiplication is four?* for the numbers sought, put x and $4 - x$, and you will have $4x - xx = 4$; and changing the signs, $xx - 4x = -4$; and compleating the square, $xx - 4x + 4 = 0$; and extracting the square root, $x - 2 = \pm 0$; whence $x = 2$, or 2, for the roots of this equation are equal; therefore 2 and 2 are the numbers desired in the question; and they will answer the conditions; for in the first place, $2 + 2 = 4 = 2 \times 2$; and in the next place, 4 the sum of 2 and 2, being added to 8, the sum of their squares, gives 12.

COROLLARY.

From our first attempt to solve this problem we may learn thus much however, that if any number whatever be made equal to y , then these two numbers y and $\frac{y}{y-1}$ will always have this property, that their sum when added together will be equal to their product when multiplied together; thus if $3 = y$, and consequently $\frac{3}{2} = \frac{y}{y-1}$, we shall have $3 + \frac{3}{2} = 4\frac{1}{2}$, and $3 \times \frac{3}{2}$ or $\frac{9}{2} = 4\frac{1}{2}$; whence it follows, that this problem cannot be solved in whole numbers in any other case than that we have here put.

PROBLEM 90.

133. *What two numbers are those, whose sum added to the product of their multiplication makes thirtyfour, and the same sum subtracted from the sum of their squares leaves fortytwo?*

SOLUTION.

Here to avoid all difficulties that would otherwise arise, put z for the sum of the two numbers sought; then since this sum added to the product of their multiplication makes 34, the product of their multiplication will be $34 - z$; but this sum z subtracted from the sum of their squares, leaves 42; therefore the sum of their squares is $42 + z$; to this add their double product $68 - 2z$, and you will have $110 - z = z^2$; whence $z = +10$, &c, and $34 - z = 24$; therefore now the question is, *What two numbers are those, whose sum is ten, and the product of their multiplication twentyfour?* and by art. III the two numbers sought are 4 and 6.

Whoever would see more questions of this nature, may consult *Bachet's* comment upon the 33d question of the first book of *Diophantus's* Arithmetics.

N. B. Having now done with quadratic equations, at least for a time, it may perhaps be expected, that according to order of method I should proceed on to equations of higher forms: but I shall take the liberty for once to dispence with that method; not but that I intend (God willing) to treat fully and distinctly of these equations hereafter; but in the mean time I think it more adviseable to employ the reader's thoughts in some other things, which I take to be of much greater importance, and more proper for his information.

THE

T H E

ELEMENTS of ALGEBRA

B O O K IV.

Of general problems, and general theorems deduced from them; together with the manner of applying, and demonstrating these theorems synthetically.

The design of this fourth book more fully explained.

134. **H**ITHERTO my young Analyst has been indulged for the most part in a sort of mixt Algebra, where letters were put only for unknown quantities: but if he would reason abstractedly upon his problems, and draw general conclusions from them, he must put letters not only for his unknown quantities, but also for such as are known; and so propose and solve his problems indefinitely. By this means in the first place he will obtain indefinite answers, which in many cases are much preferable to more particular ones, as they suit and solve all particular cases to which they are applicable; and in the next place he will be able to prove his work synthetically; which will not only confirm his former *analysis*, but will also further enure and reconcile him to the operations of symbolical or specious Arithmetic; and so render him entire master of this sort of computation. A sufficient specimen of this sort of reasoning both in the analytical and synthetical way, has already been given in our general theorem for the resolution of a quadratic equation, so that no more needs be said by way of preparation; it remains therefore now, that we look back upon some of the problems already solved, and shew how to solve them over again in general terms, as follows.

P R O B.

PROBLEM I. (See art. 26.)

135. *What two numbers are those, whose sum is a , and difference b ?*

SOLUTION.

Put x for the less number; then will the greater be $x + b$, and their sum $2x + b = a$; whence $2x = a - b$, and x (the less number) will be $\frac{a-b}{2}$; whence $x + b$ (the greater number) will be $\frac{a-b}{2} + \frac{b}{1} = \frac{a-b+2b}{2} = \frac{a+b}{2}$; so the greater number is found to be $\frac{a+b}{2}$, and the less $\frac{a-b}{2}$; where a and b are left undetermined till some particular case of this problem is proposed to be compared with the general one; and then the quantities a and b will not only be determined in that case, but the problem may be solved by the general theorem without any further *analysis*. As for example, let it be proposed, as in art. 26, to find two numbers whose sum is 48, and difference 14: here it is plain that a in the general problem answers to 48 in the particular case, and b to 14; whence $\frac{a+b}{2}$ (or the greater number) $= \frac{48+14}{2} = \frac{62}{2} = 31$, and $\frac{a-b}{2}$ (or the less number) $= \frac{48-14}{2} = \frac{34}{2} = 17$; so that the numbers sought are 31 and 17; which will answer the conditions of the question. Again, suppose we were to find two numbers whose sum is 35, and whose difference is 9: in this case it is plain that a and b have other significations; for here $a = 35$, and $b = 9$, and therefore $\frac{a+b}{2}$ (or the greater number) will be 22, and $\frac{a-b}{2}$ (or the less number) will be 13.

These theorems are capable of being translated out of Algebraic language into any other; though to no great purpose that I know of, to such as understand any thing of symbolical Arithmetic; for in my opinion, they appear much more distinct as they are, and less liable to ambiguity. The foregoing problem, together with the answer belonging to it, being translated into common English, will stand thus:

PROBLEM.

It is required, having given the sum and difference of any two numbers, to find the numbers themselves.

Ans.

Art. 135, 136. AND THEOREMS DEDUCED FROM THEM. 215

Anf. 1st, *Add the difference to the sum, and half the aggregate will be the greater number.* 2dly, *Subtract the difference from the sum, and half the remainder will be the less number.*

That this is a true translation is plain: for what is $\frac{a+b}{2}$ but half the aggregate of the sum and difference added together? and what is $\frac{a-b}{2}$ but half the remainder, after the difference is subtracted from the sum?

We come now in the last place to examine this theorem as it stands in general terms, and to try whether it will answer the conditions of the problem in the letters themselves. It was proposed to find two numbers, whose sum is a , and whose difference is b ; and the answer was, that the greater number was $\frac{a+b}{2}$, and the less $\frac{a-b}{2}$: now that this is a true answer, will be evident from a bare addition and subtraction of the numbers themselves, without any other principles; for if $\frac{a+b}{2}$ be added to $\frac{a-b}{2}$, their sum will be $\frac{2a}{2}$ or a , which answers the first condition of the problem; and if $\frac{a-b}{2}$ be subtracted from $\frac{a+b}{2}$, the remainder will be $\frac{2b}{2}$ or b , which answers the second condition.

This is that which is called a *synthetical demonstration*, and doubtless shews the truth of the theorem to which it belongs, as well as the *analysis* whereby that theorem was investigated; but not so much to the satisfaction of the mind: for a *synthetical demonstration* only shews that a proposition is true; whereas an *analytical* one shews not only that a proposition is true, but why it is so, places you in the condition of the inventor himself, and unveils the whole mystery. *Synthetical demonstrations* usually require fewer principles than *analytical* ones, as will evidently appear, by comparing both, in this very example; and this I take to be the reason why the ancients, generally speaking, chose to demonstrate their propositions this way; not with a design to conceal their *analysis*, as some have, unjustly enough, imagined; but because this sort of demonstration required fewer principles to proceed upon, and those too, such as were commonly known.

PROBLEM 2.

136. *What three numbers are those; whereof the sum of the first and second is a , that of the first and third b , and that of the second and third c ?*

S O L U -

SOLUTION.

Put x for the first number sought; then will the second number be $a-x$, because the first and second numbers together make a ; for a like reason the third number will be $b-x$, because the first and third together make b : add now the second and third numbers together, and you will have $a+b-2x=c$; therefore $2x+c=a+b$; therefore $2x=a+b-c$; and x (or the first number) $= \frac{a+b-c}{2}$; subtract now this first number $\frac{a+b-c}{2}$ from a , or which is all one, add $\frac{-a-b+c}{2}$ to a , and you will have the second number equal to $\frac{-a-b+c}{2} + \frac{a}{1} = \frac{-a-b+c+2a}{2} = \frac{a-b+c}{2}$; again, subtract the first number $\frac{a+b-c}{2}$ from b , and you will have the third number equal to $\frac{-a-b+c}{2} + \frac{b}{1} = \frac{-a+b+c}{2}$; and thus we have all the three numbers sought,

to wit,

The first,	$\frac{a+b-c}{2},$
The second,	$\frac{a-b+c}{2},$
The third,	$\frac{-a+b+c}{2}.$

To apply this general solution to some particular case, I shall make use of that in art. 42, where it was required to find three such numbers, that the sum of the first and second may make 60, that of the first and third 80, and that of the second and third 92: in this case it is plain that $a=60$, $b=80$, and $c=92$; therefore $\frac{a+b-c}{2}$ or the first number will be 24; $\frac{a-b+c}{2}$ or the second number will be 36; and $\frac{-a+b+c}{2}$ or the third number will be 56; which numbers upon trial will be found to be such as the problem requires. But that the theorems here given are not only true in this particular case, but are universally so, will best appear from the synthetical demonstration following.

1st,

Art. 136. AND THEOREMS DEDUCED FROM THEM. 217

1st, The first number $\frac{a+b-c}{2}$, and the second number $\frac{a-b+c}{2}$ being added together make $\frac{2a}{2}$ or a , according to the first condition, the other quantities destroying one another.

2dly, The first number $\frac{a+b-c}{2}$, and the third number $\frac{-a+b+c}{2}$ being added together make $\frac{2b}{2}$ or b , according to the second condition.

Lastly, The second number $\frac{a-b+c}{2}$ and the third number $\frac{-a+b+c}{2}$ being added together make $\frac{2c}{2}$ or c , according to the third condition.

This problem may also be solved somewhat more elegantly thus: put s for the unknown sum of all the three numbers sought; then if c , the sum of the second and third numbers be subtracted from s , the sum of all three, there will remain the first number equal to $s-c$; in like manner b , the sum of the first and third numbers, subtracted from s , the sum of all three, leaves the second number equal to $s-b$; and a , the sum of the first and second numbers, subtracted from s , the sum of all three, leaves the third number equal to $s-a$; add now all these three numbers together, to wit, $s-c$, $s-b$ and $s-a$, and the sum will be $3s-a-b-c$; but the sum is s , by the supposition; therefore $3s-a-b-c=s$; and $s=\frac{a+b+c}{2}$; whence we have the following theorem:

Make $\frac{a+b+c}{2}=s$; then if the numbers a , b and c be taken backwards, and subtracted severally from s , the three remainders $s-c$, $s-b$ and $s-a$ will be the three numbers sought, in order as they are supposed in the problem. Thus if $a=60$, $b=80$, and $c=92$, as before, we shall have $\frac{a+b+c}{2}$ or $s=116$; whence the first number will be $116-92$ or 24 , the second $116-80$ or 36 , and the third $116-60$ or 56 .

SCHOLIUM.

What three numbers are those, whereof the product of the first and second is a , that of the first and third b , and that of the second and third c ?

E e

SOLU-

SOLUTION.

Put p for the product of all the three numbers; then since c is the product of the two last, we shall have the first number equal to $\frac{p}{c}$; for a like reason the second equals $\frac{p}{b}$, and the third equals $\frac{p}{a}$, and the product of all three equals $\frac{p^3}{abc} = p$; therefore $p^2 = abc$, and $p = \sqrt{abc}$.

DEMONSTRATION.

$\frac{p}{c} \times \frac{p}{b}$, or the product of the first and second numbers, is $\frac{p^2}{bc} = \frac{abc}{bc} = a$; and so of the rest.

PROBLEM 3.

137. *It is required to find two numbers whose difference is b , and the difference of whose squares is a .*

SOLUTION.

Put x for the less number, and consequently $x+b$ for the greater; then will the square of the less number be xx , that of the greater $xx+2bx+bb$, and the difference of their squares $2bx+bb=a$; therefore $2bx=a-bb$, and x (the less number) $= \frac{a-bb}{2b}$; whence $x+b$ (the greater) $= \frac{a-bb}{2b} + \frac{b}{1} = \frac{a-bb+2bb}{2b} = \frac{a+bb}{2b}$.

To apply this general solution, let it be required to find two numbers whose difference is 4, and the difference of whose squares is 112: here $a=112$, $b=4$, $bb=16$, $\frac{a-bb}{2b} = 12$, $\frac{a+bb}{2b} = 16$; therefore the numbers are 12 and 16. The general demonstration is as follows: if the less number $\frac{a-bb}{2b}$ be subtracted from the greater $\frac{a+bb}{2b}$, their difference will be $\frac{2bb}{2b}$ or b , according to the first condition of the problem; again, the square of the less number $\frac{a-bb}{2b}$ is $\frac{aa-2abb+b^4}{4bb}$, and the square of the greater $\frac{a+bb}{2b}$ is $\frac{aa+2abb+b^4}{4bb}$; subtract the square of the less

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less from that of the greater, and you will have the difference of their

squares $= \frac{4abb}{4bb} = a$, as the second condition requires.

PROBLEM 4.

138. Let r and s be two given multipliers, whereof r is the greater; it is required to divide a given number as a into two such parts, that the greater part when multiplied into the less multiplier may be equal to the less part when multiplied by the greater multiplier.

SOLUTION.

Put x for the greater part, and $a-x$ for the less; then will the greater part multiplied into the less multiplier be sx , and the less part multiplied into the greater multiplier will be $ar-rx$; but according to the problem, these products are to be equal; therefore $sx=ar-rx$, and $rx+sx=ar$; but $rx+sx$ is $x \times r+s$; therefore $x \times r+s=ar$; and x (the greater of the two parts sought) $= \frac{ar}{r+s}$; whence $a-x$, (the less part) equals $\frac{a}{1} - \frac{ar}{r+s} = \frac{ar+as-ar}{r+s} = \frac{as}{r+s}$; so the greater part sought is $\frac{ar}{r+s}$, and the less $\frac{as}{r+s}$.

THE APPLICATION.

To apply this canon, let it be required to divide 84 into two such parts, that five times one part may be equal to seven times the other: here

$a=84$, r the greater multiplier $=7$, $s=5$, $\frac{ar}{r+s} = \frac{7 \times 84}{12} = 49$,

$\frac{as}{r+s} = \frac{5 \times 84}{12} = 35$; therefore the greater part is 49, and the less 35;

and they will answer the conditions; for first, $49+35=84$; and secondly, $49 \times 5 = 245 = 35 \times 7$. Again, let it be required to divide 99 into two such parts, that $\frac{2}{3}$ of one part may be equal to $\frac{4}{5}$ of the other:

here $a=99$, $r=\frac{4}{5}$, $s=\frac{2}{3}$, $r+s=\frac{22}{15}$, $\frac{r}{r+s} = \frac{\frac{4}{5}}{\frac{22}{15}} = \frac{6}{11}$, $\frac{s}{r+s} = \frac{\frac{2}{3}}{\frac{22}{15}} = \frac{5}{11}$,

$\frac{ar}{r+s} = 99 \times \frac{6}{11} = 54$, $\frac{as}{r+s} = 99 \times \frac{5}{11} = 45$; so the two parts are 54 and 45; which is true; for first, $54+45=99$; and secondly, $\frac{2}{3}$ of 54 $= 36 = \frac{4}{5}$ of 45.

As to the demonstration of this general solution, it must be observed that in this problem there are two conditions; first, that the two parts when added together must make a ; and secondly, that the greater part multiplied into the less multiplier must be equal to the less part multiplied into the greater multiplier: as to the first of the conditions, it is certain that the parts $\frac{ar}{r+s}$ and $\frac{as}{r+s}$ when added together will make

$$\frac{ar+as}{r+s}; \text{ but } ar+as = a \times r+s, \text{ therefore } \frac{ar+as}{r+s} = a \times \frac{r+s}{r+s} = a$$

$\times 1 = a$: as to the second condition, if the greater part $\frac{ar}{r+s}$ be multi-

plied into s , the less multiplier, the product will be $\frac{ars}{r+s}$; and again,

if the less part $\frac{as}{r+s}$ be multiplied into r , the greater multiplier, the

product will also be $\frac{ars}{r+s}$; therefore the two products are equal, as the problem requires; and so the conditions are both satisfied. *Q. E. D.*

N. B. If any one has a mind to throw the foregoing theorem into words, it may easily be done, and in such a manner as almost to carry it's own evidence along with it; for by the rule of proportion, $r+s$ is to r as a to $\frac{ar}{r+s}$; and $r+s$ is to s as a to $\frac{as}{r+s}$; therefore, *As the sum of the two multipliers is to the greater or less multiplier, so is the sum of the two parts sought to the greater or less part*: and this, I say, is pretty evident; for had $r+s$ been the number to be divided, the parts would certainly have been r and s ; therefore if a greater or less number than $r+s$ is to be divided, the parts ought to be greater or less than r and s in the same proportion.

PROBLEM 5.

139. *Let r and s be two given multipliers whereof r is the greater; it is required to divide a given number as a into two such parts, that r times one part being added to s times the other may make some other given number, as b .*

SOLUTION.

Put x for the part that is to be multiplied by r , and consequently $a-x$ for the other part that is to be multiplied by s , and the products will

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will be rx and $as - sx$, and their sum will be $rx + as - sx = b$; therefore $rx - sx = b - as$, that is, $x \times r - s = b - as$; therefore x (the part to be multiplied by r) $= \frac{b - as}{r - s}$; therefore $a - x$ (the part to be multiplied by s) $= \frac{a - b + as}{r - s} = \frac{ar - as - b + as}{r - s} = \frac{ar - b}{r - s}$.

The APPLICATION.

Let it be required to divide 20 into two such parts, that three times one part being added to five times the other may make 84: here $a = 20$, $b = 84$, $r = 5$, $s = 3$, $as = 60$, $b - as = 24$, $\frac{b - as}{r - s}$ (or the part to be multiplied by 5) $= \frac{24}{2} = 12$, $ar = 100$, $ar - b = 16$, $\frac{ar - b}{r - s}$ (or the part to be multiplied by 3) $= \frac{16}{2} = 8$; therefore the parts sought are 8 and 12; for first, $8 + 12 = 20$; and secondly, three times 8 + five times 12 = 84.

Again, let it be required to divide 100 into two such parts, that $\frac{3}{4}$ of one part being subtracted from $\frac{5}{6}$ of the other, may leave 39: here it must be observed, that to subtract $\frac{3}{4}$ of any one quantity from another, is the same as to add $\frac{-3}{4}$ of it; therefore this problem when reduced to the form of the general one, will stand thus: *To divide a hundred into two such parts, that $\frac{-3}{4}$ of one part being added to $+\frac{5}{6}$ of the other may make*

$$\begin{aligned} \text{thirtynine. Here } a &= 100, b = 39, r = \frac{5}{6}, s = \frac{-3}{4}, r - s = \frac{5}{6} + \frac{3}{4} \\ &= \frac{19}{12}, as = \frac{-300}{4} = -75, b - as = 39 + 75 = 114, \frac{b - as}{r - s} = \frac{114}{\frac{19}{12}} \\ &= 72, ar = \frac{500}{6} = \frac{250}{3}, ar - b = \frac{250}{3} - \frac{39}{1} = \frac{133}{3}, \frac{ar - b}{r - s} = \frac{\frac{133}{3}}{\frac{19}{12}} \\ &= 28; \text{ so the two parts are 28 and 72; for } 28 + 72 = 100; \text{ and moreover } \frac{3}{4} \text{ of 28, that is, 21, subtracted from } \frac{5}{6} \text{ of 72, that is, from 60, leaves 39.} \end{aligned}$$

The GENERAL DEMONSTRATION.

The two parts $\frac{ar - b}{r - s}$ and $\frac{b - as}{r - s}$ when added together, make

$$ar - b$$

$\frac{ar-b+b-as}{r-s} = \frac{ar-as}{r-s} = a \times \frac{r-s}{r-s} = a$: again, the part $\frac{b-as}{r-s}$ being multiplied into r , it's proper multiplier, gives $\frac{br-ars}{r-s}$, and the other part $\frac{ar-b}{r-s}$, multiplied into the other multiplier s , gives $\frac{ars-bs}{r-s}$; add these two products together, and they will make $\frac{br-ars+ars-bs}{r-s} = \frac{br-bs}{r-s} = b$. Q. E. D.

If any one hereafter shall think me too concise in the solutions of these general problems, he must have recourse to the particular ones in the articles I shall refer him to, which he will find explained more at large: and as to the application of these general solutions to those particular cases, it is to be presumed that by this time the learner will be able in some measure to perform that part himself; and therefore I shall for the future leave it to him, except where I shall think my assistance may be of any use.

PROBLEM 6. (See art. 35.)

140. One meeting a company of beggars, gives to each p pence, and has a pence over; but if he would have given them q pence apiece, he would have found he had wanted b pence for that purpose: What was the number of persons?

SOLUTION.

The number of persons, x .

Pence given, px .

Pence in all, $px+a$.

The pence that would have been given upon the other supposition, qx .

Another expression for the number of pence in all, $qx-b$.

Equ. $qx-b=px+a$; therefore $qx-px-b=a$; therefore $qx-px=a+b$; therefore x (the number of persons) $= \frac{a+b}{q-p}$.

DEMONSTRATION.

If the number of persons be $\frac{a+b}{q-p}$, then the pence given will be $\frac{ap+bp}{q-p}$, and the pence in all will be $\frac{ap+bp}{q-p} + \frac{a}{1} = \frac{ap+bp+aq-ap}{q-p} =$

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$= \frac{aq+bp}{q-p}$: again, the number of pence that would have been given upon the second supposition is $\frac{aq+bq}{q-p}$; and therefore the other expression for the number of pence in all will be $\frac{aq+bq}{q-p} - \frac{b}{1} = \frac{aq+bp}{q-p}$; and the perfect agreement between this account and the former, is an infallible argument that the number of persons was rightly assigned.

PROBLEM 7. (See art. 64.)

141. *It is required to divide a given number as a into two such parts, that one part may be to the other as r to s.*

SOLUTION.

The two parts sought, x and $a-x$.

Proportion, x is to $a-x$ as r to s .

Equation, $sx = ar - rx$; therefore $rx + sx = ar$; therefore x (or the first number) $= \frac{ar}{r+s}$; therefore $a-x$ (or the second number) $= \frac{a}{1} - \frac{ar}{r+s} = \frac{as}{r+s}$; therefore the two numbers are $\frac{ar}{r+s}$ and $\frac{as}{r+s}$.

DEMONSTRATION.

1st, The two numbers $\frac{ar}{r+s}$ and $\frac{as}{r+s}$ when added together make $\frac{ar+as}{r+s} = a$.

2dly, The first number $\frac{ar}{r+s}$ is to the second number $\frac{as}{r+s}$ as ar is to as ; because throwing away the common denominator is no more in reality than multiplying both fractions by it; and every one knows, that the multiplication of two quantities by the same number, makes no alteration in the proportion they bore one to the other; again, ar is to as (dividing both by a) as r to s ; for it is as well known that a common division affects proportion no more than a common multiplication: since then the first number is to the second as ar to as , and ar is to as as r to s , it follows, that the first number is to the second as r to s . Q. E. D.

PRO-

PROBLEM 8. (See art. 66.)

142. *What number is that, which being severally added to two given numbers, a a greater number, and b a less, will make the former sum to the latter as r to s? therefore r must be greater than s.*

SOLUTION.

The number sought, x .

Proportion, $a+x$ is to $b+x$ as r to s .

Equation, $br+rx=as+sx$; therefore $br+rx-sx=as$; therefore $rx-sx=as-br$; therefore $x=\frac{as-br}{r-s}$.

DEMONSTRATION.

The number $\frac{as-br}{r-s}$ being added to a , gives $\frac{ar-br}{r-s}$; and the same number being added to b , gives $\frac{as-bs}{r-s}$; now $\frac{ar-br}{r-s}$ is to $\frac{as-bs}{r-s}$ as $ar-br$ is to $as-bs$, that is, as $r \times a - b$ is to $s \times a - b$, that is, as r to s . Q. E. D.

SCHOLIUM.

This problem was to find a number, which being severally added to a and b , will make the former sum to the latter as r to s ; let us now change the numbers a and b one for another, as also the numbers r and s one for another, and then the problem will stand thus: *To find a number which being severally added to b and a , will make the former sum to the latter as s to r* : but the condition of this problem is exactly the same with that of the former, and therefore the answer ought still to be the same; that is, as changing a and b one for another, and r and s one for another, had no effect upon the problem, but left it entirely the same as at first; so if the expression of the number sought be just, the changing of a and b one for another, and of r and s one for another, ought to make no alteration in that expression, and the number sought ought still to be the same; for truth will always be consistent with herself. Let us try this however, and see what will be the effect of such a change:

now the number sought was $\frac{as-br}{r-s}$; but upon this change, as becomes br , and br becomes as , and $r-s$ becomes $s-r$, and the whole expression will be turned into this, $\frac{br-as}{s-r}$; but $\frac{br-as}{s-r}$ is the same

as

as $\frac{as-br}{r-s}$; for changing the sign of both the numerator and denominator of any fraction, no more affects the value of that fraction, than in division the changing of the sign both of the divisor and dividend affects the value of the quotient: thus then we find, that the changing of a and b one for another and of r and s one for another, no more affects the theorem for determining the number sought, than it did the problem from whence it was derived.

PROBLEM 9.

143. *It is required to divide a given number as a into two such parts, that the excess of one part above another given number as b , may be to what the other wants of b , as r to s ; supposing r greater than s .*

SOLUTION.

Put x for the greater part, and $a-x$ for the less; then the excess of x above b will be $x-b$; and the excess of b above $a-x$ will be $x-a+b$, as appears by subtracting $a-x$ from b ; but by the problem, the former excess is to the latter as r to s ; therefore $x-b$ is to $x-a+b$ as r to s ; multiply extremes and means, and you will have $sx-bs = rx-ar+br$; therefore $rx-sx = ar-br-bs$, and x (the greater part) $= \frac{ar-br-bs}{r-s}$; therefore $a-x$ (the less part) $= \frac{a}{1} - \frac{ar-br-bs}{r-s} = \frac{br+bs-as}{r-s}$; so the greater part is $\frac{ar-br-bs}{r-s}$, and the less part $\frac{br+bs-as}{r-s}$.

EXAMPLE.

Let it be required (as in art. 41,) to divide the number 48 into two such parts, that one part may be three times as much above 20 as the other wants of 20: here $a=48$, $b=20$, $r=3$, $s=1$; for to say that the excess must be three times the defect, is no other than to say, that the excess must be to the defect as 3 to 1; the rest is easy.

The GENERAL DEMONSTRATION.

1st, The greater part $\frac{ar-br-bs}{r-s}$, and the less part $\frac{br+bs-as}{r-s}$
F f
being

being added together make $\frac{ar-as}{r-s} = a$: again, the excess of the greater part above b , is $\frac{ar-br-bs}{r-s} - \frac{b}{1} = \frac{ar-br-bs-br+bs}{r-s} = \frac{ar-2br}{r-s}$, and the excess of b above the less part, which is what the less part wants of b , is $\frac{b}{1} - \frac{br-bs+as}{r-s} = \frac{br-bs-br-bs+as}{r-s} = \frac{as-2bs}{r-s}$; therefore the excess of one part above b is to what the other wants of b , as $\frac{ar-2br}{r-s}$ is to $\frac{as-2bs}{r-s}$, that is, as $ar-2br$ is to $as-2bs$, that is, as $r \times a - 2b$ is to $s \times a - 2b$, or as r to s . Q. E. D.

PROBLEM 10. (See art. 55.)

144. *There are two places whose distance from each other is a , and from whence two persons set out at the same time with a design to meet, one travelling at the rate of p miles in q hours, and the other at the rate of r miles in s hours: I demand how long and how far each travelled before they met.*

SOLUTION.

The number of hours travelled by each, x .

Miles travelled by the first, $\frac{px}{q}$.

By the second, $\frac{rx}{s}$.

By them both, $\frac{px}{q} + \frac{rx}{s}$.

Equation, $\frac{px}{q} + \frac{rx}{s} = a$; therefore $px + \frac{qrx}{s} = aq$; therefore $psx + qrx = aqs$; therefore x (or the number of hours travelled by each) $= \frac{aqs}{ps+qr}$: now to find how many miles the first travelled, say, if in q hours he travelled p miles, how many will he travel in a number of hours equal to $\frac{aqs}{ps+qr}$? for a fourth number, I multiply the third number $\frac{aqs}{ps+qr}$ by the second p , and the product is $\frac{apqs}{ps+qr}$; this again I divide

divide by the first number q , and the quotient is $\frac{aps}{ps+qr}$; for dividing the numerator divides the whole fraction: by the same way of reasoning, the number of miles travelled by the other will be found to be $\frac{aqr}{ps+qr}$: therefore the whole number of miles travelled by them both is $\frac{aps+aqr}{ps+qr} = a$, which demonstrates the solution.

E X A M P L E.

Let the distance of the two places be 154 miles; let the first travel at the rate of 3 miles in 2 hours, and the second after the rate of 5 miles in 4 hours; then we shall have $a=154$, $p=3$, $q=2$, $r=5$, $s=4$, $ps=12$, $qr=10$, $ps+qr=22$, $\frac{aqs}{ps+qr} = \frac{154 \times 2 \times 4}{22} = 56$, $\frac{aps}{ps+qr} = \frac{154 \times 3 \times 4}{22} = 84$, $\frac{aqr}{ps+qr} = \frac{154 \times 2 \times 5}{22} = 70$: therefore each travelled 56 hours; the first travelled 84 miles, and the other 70.

S C H O L I U M.

If in the foregoing problem we change p into r and q into s , and *vice versa*, the consequence will be, that the first traveller will now travel at the same rate as the second did before, and the second at the same rate as the first did before; but the motion whereby these two travellers approach towards each other will still be the same, and therefore the time this motion is performed in, that is, the time that each travelled, must still be the same: let us then make the changes abovementioned, first in the expression of the time, and see whether that expression will still continue the same; then let us make the same changes in the two expressions of the miles, and see whether by this means, these expressions will not be converted each into the other: first then, the expression of the time, which is $\frac{aqs}{ps+qr}$, by changing p into r , and q into s , and *vice versa*, becomes $\frac{asq}{rq+sp}$, which is the same as $\frac{aqs}{ps+qr}$; therefore the expression of the time suffers no alteration by these changes: secondly, the number of miles travelled by the first was $\frac{aps}{ps+qr}$, which after the changes abovementioned, becomes $\frac{arq}{rq+sp}$, which is the same

as $\frac{aqr}{ps+qr}$, the miles travelled by the second; and therefore *à converso*, the expression $\frac{aqr}{ps+qr}$ will be changed into the expression $\frac{aps}{ps+qr}$; and thus will the case of the first traveller be changed into that of the second, and *vice versa*.

PROBLEM II. (See art. 46.)

145. Suppose that p pounds of gold out of water weigh q pounds in water, and that r pounds of silver weigh s pounds in water; suppose also that a mass weighing a pounds, and consisting of both gold and silver, when weighed in water, weighs only b pounds: I demand the distinct quantities of gold and silver in the mass.

SOLUTION.

The number of pounds of gold in the mass, x .

Of silver, $a-x$.

The weight of the former in water, $\frac{qx}{p}$.

Of the latter, $\frac{as-sx}{r}$.

Equation, $\frac{qx}{p} + \frac{as-sx}{r} = b$; therefore $qx + \frac{aps-psx}{r} = bpr$; therefore $qrx + aps - psx = bpr$; therefore $qrx - psx = bpr - aps$; therefore x (or the number of pounds of gold in the mass) $= \frac{bpr - aps}{qr - ps}$; therefore $a-x$ (or the pounds of silver) $= \frac{a}{1} - \frac{bpr + aps}{qr - ps} = \frac{aqr - bpr}{qr - ps}$.

DEMONSTRATION.

First, the weight of the gold $\frac{bpr - aps}{qr - ps}$, and the weight of the silver $\frac{aqr - bpr}{qr - ps}$ being added together make $\frac{aqr - aps}{qr - ps} = a$.

2dly, We must find in the next place, how much each weighs in water, by saying, if p pounds of gold weigh q pounds in water, what will $\frac{bpr - aps}{qr - ps}$ weigh? and the answer will be $\frac{bqr - aqs}{qr - ps}$; in like manner,

the

the weight of the silver in water will be found to be $\frac{aqs - bps}{qr - ps}$.

3dly, Add both these weights together, that is, $\frac{bqr - aqs}{qr - ps}$, and $\frac{aqs - bps}{qr - ps}$, and they will make $\frac{bqr - bps}{qr - ps}$, or *b. Q. E. D.*

PROBLEM 12.

146. *There are two pipes, whereof one will fill a cistern in the time p, and the other in the time q: In what time will they both fill it?*

SOLUTION.

Put x for the time sought; then say, if in the time p the first pipe will discharge one cisternful, how much will it discharge in the time x ? and the answer will be $\frac{x}{p}$; and for the same reason, $\frac{x}{q}$ will represent the quantity discharged by the other pipe in the same time x ; therefore $\frac{x}{p} + \frac{x}{q}$ will be the quantity discharged by them both; but by the problem, they ought both to discharge one cisternful in this time; therefore we have this equation, $\frac{x}{p} + \frac{x}{q} = 1$; therefore $x + \frac{px}{q} = p$; therefore $qx + px = pq$; therefore x (or the time sought) $= \frac{pq}{p+q}$.

DEMONSTRATION.

1st, If in the time p the first pipe discharges one cisternful, in the time $\frac{pq}{p+q}$ it will discharge the quantity $\frac{q}{p+q}$.

2dly, And for the same reason, in the same time $\frac{pq}{p+q}$, the other pipe will discharge the quantity $\frac{p}{p+q}$.

3dly, Therefore in the time $\frac{pq}{p+q}$, both pipes together will discharge the quantity $\frac{p+q}{p+q} = 1$, that is, one cisternful. *Q. E. D.*

PRO-

PROBLEM 13.

147. One has a certain number n , of children whose ages are in arithmetical progression; the common difference of the progression is d , and the age of the eldest child is to that of the youngest as r to s : What are the ages of the eldest and youngest?

SOLUTION.

Here I might put x for the age of the eldest child, and consequently $x-d$ for the age of the second, and so on; but the work will succeed better, if we put $x-d$ or $x-1d$ for the age of the first child, $x-2d$ for the age of the second, $x-3d$ for the age of the third, $x-4d$ for the age of the fourth, &c; for by this means, the number of every child, reckoning from the first inclusively, will also be the coefficient of d in the expression of his age; but according to this way of reckoning, the number of the last, or youngest child will be n ; therefore his age will be $x-nd$; whence we have this proportion, $x-d$ is to $x-nd$ as r to s ; and this equation, $rx-dnr=sx-ds$; therefore $rx-sx=dnr-ds$; therefore $rx-sx=dnr-ds$; therefore $x=\frac{dnr-ds}{r-s}$; therefore $x-d$ (or the age of the eldest child) is $\frac{dnr-ds}{r-s}-\frac{d}{1}=\frac{dnr-dr}{r-s}$; and $x-nd$ (or the age of the youngest) is $\frac{dnr-ds}{r-s}-\frac{nd}{1}=\frac{dns-ds}{r-s}$.

DEMONSTRATION.

According to this computation, the age of the eldest is to that of the youngest as $\frac{dnr-dr}{r-s}$ is to $\frac{dns-ds}{r-s}$, that is, as $dnr-dr$ is to $dns-ds$, that is, as $nr-r$ is to $ns-s$, that is, as $r \times n-1$ is to $s \times n-1$, that is, as r to s . Q. E. D.

PROBLEM 14. (See art. 68.)

148. What number is that, which being severally added to a , b and c , all given numbers, will make the three sums to be in continual geometrical proportion? where the reader may, if he pleases, suppose a to be the greatest, and c the least of the three numbers given.

SOLU-

SOLUTION.

Put x for the number sought; and then the proportion will stand thus,

$x + a$ is to $x + b$ as $x + b$ is to $x + c$.

Multiply extremes and means, and you will have this equation, $xx + ax + cx + ac = xx + 2bx + bb$; throw away xx from both sides, and you will have $ax + cx + ac = 2bx + bb$; therefore $ax - 2bx + cx + ac = bb$; therefore $ax - 2bx + cx = bb - ac$; therefore x (or the number sought) $= \frac{bb - ac}{a - 2b + c}$.

DEMONSTRATION.

1st, If the number sought $\frac{bb - ac}{a - 2b + c}$ be added to the first given number a , the sum will be $\frac{aa - 2ab + bb}{a - 2b + c} = \frac{a - b \times a - b}{a - 2b + c}$, which I shall therefore call the first extreme.

2dly, If $\frac{bb - ac}{a - 2b + c}$ be added to the second number b , the sum will amount to $\frac{bb - ac + ab - 2bb + bc}{a - 2b + c} = \frac{ab - bb - ac + bc}{a - 2b + c}$; but for the better conceiving of this fraction, I divide the numerator into two parts, viz. $ab - bb$, and $-ac + bc$; then it is plain that the first part $ab - bb$ is the product of $a - b \times b$, and the second part $-ac + bc$ is the product of $a - b \times -c$; therefore the whole numerator is the product of $a - b \times b - c$, and the middle term is $\frac{a - b \times b - c}{a - 2b + c}$.

3dly, $\frac{bb - ac}{a - 2b + c}$ added to the third given number c , makes $\frac{bb - 2bc + cc}{a - 2b + c} = \frac{b - c \times b - c}{a - 2b + c}$, which therefore will be the last extreme.

4thly, We are now in the next place to enquire whether these three sums be in continual geometrical proportion: now if we compare the two first terms together, we shall find that the first sum is to the second as $\frac{a - b \times a - b}{a - 2b + c}$ is to $\frac{a - b \times b - c}{a - 2b + c}$, that is, as $a - b \times a - b$ is to $a - b \times b - c$, that is, as $a - b$ is to $b - c$.

5thly,

5thly, If we compare the second and third terms together, we shall find that the second sum is to the third as $\frac{a-b \times b-c}{a-2b+c}$ is to $\frac{b-c \times b-c}{a-2b+c}$, that is, as $a-b \times b-c$ is to $b-c \times b-c$, that is, as $a-b$ is to $b-c$.

6thly, Since then the first sum is to the second as $a-b$ is to $b-c$, and the second sum is to the third also as $a-b$ is to $b-c$, it follows, that the first sum is to the second as the second is to the third, and consequently, that the three sums are in continual proportion. Q. E. D.

S C H O L I U M.

The last problem was, to find a number, which being severally added to three given ones a , b and c , will make them continual proportionals: let us now change this problem into another, by changing the extreme numbers a and c one for the other; and then the question will be, to find a number, which being severally added to c , b and a , will make them continual proportionals: here it is plain, first, that the number sought in this case, must be the same with the number sought in the former; 2dly, that the middle proportional must also be the same here as there; and lastly, that the extremes must be the same in both cases, but interchanged one for the other: now if the expressions in the former case be just, the substitution of a and c one for the other ought to have the same effect in those expressions as it had in the nature of the problem itself; let us try this, by actually changing a and c one for the other in all those expressions, and let us see what will be the consequence.

First then, the number sought in the former case was $\frac{bb-ac}{a-2b+c}$; put a for c and c for a , and then that expression will be changed into this, $\frac{bb-ca}{c-2b+a}$; but $bb-ca$ is the same as $bb-ac$, and $c-2b+a$ is the same as $a-2b+c$; therefore the number sought in this case is $\frac{bb-ac}{a-2b+c}$, the same as in the former.

2dly, The middle proportional in the former case was $\frac{a-b \times b-c}{a-2b+c}$; change a and c one for the other, and then the expression will be $\frac{c-b \times b-a}{c-2b+a}$; but $c-b \times b-a$ is the same as $b-c \times a-b$ or $a-b \times b-c$, because a change of the signs in both factors will have no more effect

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effect upon the product, than if no such change had been made; and $c - 2b + a$ is the same as $a - 2b + c$, as was before observed; therefore

the middle proportional in this case is $\frac{a - b \times b - c}{a - 2b + c}$, which is the same as in the former.

3dly, The first extreme in the former case was $\frac{a - b \times a - b}{a - 2b + c}$; put a and c for one another in this expression, and it will be changed into $\frac{c - b \times c - b}{c - 2b + a} = \frac{b - c \times b - c}{a - 2b + c}$; therefore the first extreme in this case is the same with the last in the former.

But I would not be mistaken in this *scholium*, or in any other of the like nature, as if I introduced them to confirm the solutions of the problems to which they are annexed; for those problems need no such confirmation: my chief design herein, is to shew the beauty and consistency of truth, the necessary connexion one truth has with another, and how much more clearly and distinctly this harmony is to be perceived in the mathematical sciences than in any other whatsoever; and yet after all, it is not impossible but that I may in a great measure lose my labour.

PROBLEM 15. (See art. 38.)

149. What two numbers are those, whereof the greater is to the less as p to q , and the product of their multiplication is to their sum as r to s ?

SOLUTION.

Put x for the less number, and the greater will be found by saying, as q is to p , so is x the less number to $\frac{px}{q}$ the greater; whence their sum will be $\frac{px}{q} + \frac{x}{1} = \frac{px + qx}{q}$: on the other hand, if the greater number $\frac{px}{q}$ be multiplied into x , the product will be $\frac{p x x}{q}$; therefore the product of these two numbers will be to their sum as $\frac{p x x}{q}$ is to $\frac{px + qx}{q}$, that is, as px to $p + q$; but according to the problem, the product is to the sum as r to s ; therefore px is to $p + q$ as r to s ; whence we have this equation, $psx = pr + qr$; and x (the less number

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ber sought) $= \frac{pr+qr}{ps}$; therefore $px = \frac{pr+qr}{s}$; for dividing the denominator multiplies the whole fraction; therefore $\frac{px}{q}$ (or the greater number) $= \frac{pr+qr}{qs}$.

DEMONSTRATION.

1st, The greater number is to the less as $\frac{pr+qr}{qs}$ is to $\frac{pr+qr}{ps}$; divide $pr+qr$ by itself, and the quotient will be 1; so that we may now say, that the greater number is to the less as $\frac{1}{qs}$ is to $\frac{1}{ps}$, that is, as $\frac{1}{q}$ is to $\frac{1}{p}$, that is as $\frac{p}{q}$ is to 1, that is, as p is to q .

2dly, The greater number $\frac{pr+qr}{qs}$ and the less $\frac{pr+qr}{ps}$ being added together make $\frac{pprs+pqrs+pqrs+qqrs}{pqss} = \frac{pprs+2pqrs+qqrs}{pqss}$; but $pp+2pq+qq = \overline{p+q}^2$; therefore the sum of the two numbers sought is $\frac{rs \times \overline{p+q}^2}{pqss}$.

3dly, The greater number $\frac{pr+qr}{qs}$ multiplied into the less $\frac{pr+qr}{ps}$ produces $\frac{rr \times \overline{p+q}^2}{pqss}$.

4thly, Therefore the product of the two numbers sought is to their sum as $\frac{rr \times \overline{p+q}^2}{pqss}$ is to $\frac{rs \times \overline{p+q}^2}{pqss}$, that is, as rr is to rs , or as r to s .
Q. E. D.

PROBLEM 16. (See art. 63.)

150. One draws a certain quantity of wine out of a full vessel that held a certain number of gallons equal to a ; and then recruiting the vessel with water, takes a second draught of as much wine and water together as before he did of wine; and so he goes on for four draughts one after another, always taking the same quantity at a draught, and then recruiting

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recruiting the vessel with water; insomuch that at last, the number of gallons of pure wine left in the vessel was only equal to b: I demand how much he took at every draught.

SOLUTION.

For the number of gallons left in the vessel after every draught, which was always the same, put x ; then it is plain that at every draught, the whole quantity of liquor in the vessel, whether it was pure wine, or wine and water mixt, must be diminished in the proportion of a to x ; therefore the wine must be diminished in that proportion without being recruited; whence we have the following proportions; 1st, as a is to x , so is x , the quantity of pure wine left in the vessel after the first draught, to $\frac{x x}{a}$, the wine left in the vessel after the second draught; 2^{dly}, as a is to x , so is $\frac{x x}{a}$, the wine left after the second draught, to $\frac{x^3}{a^2}$, the wine left after the third draught; lastly, as a is to x , so is $\frac{x^3}{a^2}$, the wine left after the third draught, to $\frac{x^4}{a^3}$, the wine left after the fourth draught; but according to the problem, the quantity of wine left after the fourth draught was b ; therefore $\frac{x^4}{a^3} = b$; therefore $x^4 = a^3 b$, $= a^4 \times \frac{b}{a}$; make $\frac{b}{a} = s^4$, and then we shall have $x^4 = a^4 s^4$, and x (or the quantity of liquor left in the vessel after every draught) $= a s$; whence $a - x$ (or the quantity taken at every draught) will be $a - a s = a \times 1 - s$.

EXAMPLE.

Suppose the vessel when full held 81 gallons, and that there were 16 gallons of pure wine left after the fourth draught; then we shall have $a = 81$, $b = 16$, $\frac{b}{a}$ or $s^4 = \frac{16}{81}$, $s s = \frac{4}{9}$, $s = \frac{2}{3}$, $1 - s = \frac{1}{3}$, $a \times 1 - s = 81 \times \frac{1}{3} = 27$; therefore 27 gallons were taken at every draught.

DEMONSTRATION.

Since the wine in the vessel was diminished at every draught in the proportion of a to x , that is, in the proportion of a to $a s$, or of 1 to s , it follows, that as 1 is to s , so is x or $a s$, the wine left after the first draught, to $a s s$, the wine left after the second draught; and for the same reason,

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as 1 is to s so is as^2 to as^3 , the wine left after the third draught; and lastly, as 1 is to s so is as^3 to as^4 , the wine left after the fourth draught; but $as^4 = a \times \frac{b}{a}$ by the supposition, $= b$; therefore the quantity of pure wine left in the vessel after the fourth draught was b . *Q. E. D.*

N. B. As there are but few numbers which admit of an exact square root, so there are fewer still which admit of exact roots of higher kinds; but in the course of this work, I shall shew how to extract all roots with equal facility, and to as great a degree of exactness as is generally required, to wit, by a full table of logarithms.

PROBLEM 17.

151. *What two numbers are those, the product of whose multiplication is p , and the quotient of the greater divided by the less is q ?*

SOLUTION.

Put x for the greater number, and consequently $\frac{p}{x}$ for the less; then will the quotient of the greater divided by the less be $\frac{xx}{p}$; but according to the problem, this quotient ought to be q ; therefore $\frac{xx}{p} = q$; and $xx = pq$, and x (the greater number sought) $= \sqrt{pq}$: again, since $xx = pq$, we have $\frac{pp}{xx} = \frac{pp}{pq} = \frac{p}{q}$; and $\frac{p}{x}$ (or the less number sought) $= \sqrt{\frac{p}{q}}$; so that the greater of the two numbers sought is \sqrt{pq} , and the less $\sqrt{\frac{p}{q}}$.

EXAMPLE.

Let the product of the two numbers sought be 144, and the quotient of the greater divided by the less 16; then we shall have $p = 144$, $q = 16$, $pq = 144 \times 16$, $\sqrt{pq} = 12 \times 4 = 48$; $\frac{p}{q} = \frac{144}{16}$, $\sqrt{\frac{p}{q}} = \frac{12}{4} = 3$; therefore the numbers are 48 and 3.

DEMONSTRATION.

1st, pq multiplied into $\frac{p}{q}$ gives $\frac{ppq}{q} = pp$; therefore \sqrt{pq} multiplied into $\sqrt{\frac{p}{q}}$ gives p .

2dly,

2dly, pq being divided by $\frac{p}{q}$ gives $\frac{pq}{p} = q$; therefore \sqrt{pq} being divided by $\sqrt{\frac{p}{q}}$ gives q . Q. E. D.

PROBLEM 18.

152. Let x and y be two unknown quantities, and a, b, c, d, e, f known quantities: It is required to find the values of x and y by the help of the two following equations, to wit, $ax + by = c$, and $dx + ey = f$.

SOLUTION.

Equation 1st, $ax + by = c$.

Equation 2d, $dx + ey = f$. Multiply the first equation by d , and you will have $adx + bdy = cd$; multiply the second equation by a , and you will have $adx + aey = af$; subtract the former equation thus produced from the latter, and you will have

Equ. 3d, $ae - bd \times y = af - cd$; whence

Equ. 4th, $y = \frac{af - cd}{ae - bd}$.

Therefore $by = \frac{abf - bcd}{ae - bd}$; put this value instead of by in the first

equation, and you will have $ax + \frac{abf - bcd}{ae - bd} = c$; multiply the whole

into $ae - bd$, and you will have $aae - abd \times x + abf - bcd = ace - bcd$; throw away $-bcd$ from both sides of the equation, and you will

have $aae - abd \times x + abf = ace$; divide the whole by a , and you will have $ae - bd \times x + bf = ce$; therefore $ae - bd \times x = ce - bf$; therefore

$x = \frac{ce - bf}{ae - bd}$; and so the quantities x and y are at last found to be

$x = \frac{ce - bf}{ae - bd}$ and $y = \frac{af - cd}{ae - bd}$.

I have here pursued the common method in finding out x ; but after y was discovered, x might have been more easily and more readily determined thus: suppose the two fundamental equations had been these that follow; $bx + ay = c$, and $ex + dy = f$; the consequence would have been, that x in this case would have come out the same as y in the last, and y here the same as x there; therefore changing a into b , and d into e ,

and *vice versa*, will change y into x ; now the expression of y is $\frac{af - cd}{ae - bd}$; change

change in this expression a into b , and d into e , and *vice versa*, and it

will then stand thus, $\frac{bf - ce}{bd - ae} = \frac{ce - bf}{ae - bd} = x$.

By this and many other instances that might be produced, it might easily be made appear, that these speculations are sometimes no less useful in calculation, than they are entertaining to men of taste and genius, though perhaps they may be found too subtil to be relished by the raw and narrow minds of young beginners

For the better conceiving of the foregoing expressions, dispose the quantities in the order following:

$$\begin{array}{cc} a & d \\ c & f \\ b & e \end{array}$$

First put down a and d , the two coefficients of x , one after the other in the same order as they stand in the two fundamental equations; under these, and in the same order, put down c and f , the two absolute terms; and lastly under these again, put down b and e , the two coefficients of y ; this done, compare the first couple a and d with the second c and f , multiplying the terms crosswise, and $af - cd$ will be the numerator of the value of y ; again, compare the second couple c and f with the third b and e , multiplying them in like manner, and $ce - bf$ will be the numerator of x ; lastly, compare the first pair a and d with the last b and e , multiplying them as before, and $ae - bd$ will be a common denominator to both the former numerators; so that the coefficients of x enter into the numerator of y , those of y into the numerator of x , and both into the common denominator.

EXAMPLE.

Let it be required to determine x and y by the help of the two following equations; $3x - 4y = 6$, and

$$5x - 6y = 14.$$

The terms being disposed according to the foregoing method, will stand thus;

$$\begin{array}{cc} 3 & 5 \\ 6 & 14 \\ -4 & -6 \end{array}$$

First then I compare the first pair 3 and 5 with the second 6 and 14, thus; $3 \times 14 = 42$, $5 \times 6 = 30$, and the latter product subtracted from the former leaves 12 for the numerator of y ; in the next place I compare the second pair 6 and 14 with the third -4 and -6 , thus; $6 \times -6 = -36$, $14 \times -4 = -56$, and the latter product subtracted from the former leaves $+20$, the numerator of x ; lastly, I compare the first couple 3 and 5 with the third -4 and -6 , thus; $3 \times -6 = -18$, $5 \times -4 = -20$,

Art. 152, 153. AND THEOREMS DEDUCED FROM THEM. 239

$= -20$, and the latter product subtracted from the former leaves $+2$ for a common denominator to both x and y ; so that we have $x = \frac{20}{2}$ or 10 , and $y = \frac{12}{2}$ or 6 ; which numbers 10 and 6 will answer the conditions proposed.

N. B. If any of the terms ax , by , dx , ey be wanting, the coefficients must be supplied with cyphers; as if ax be wanting, a must be supposed equal to 0 .

THE GENERAL DEMONSTRATION.

1st, Since $x = \frac{ce - bf}{ae - bd}$ and $y = \frac{af - cd}{ae - bd}$, we shall have $ax = \frac{ace - abf}{ae - bd}$, and $by = \frac{abf - bcd}{ae - bd}$; therefore $ax + by = \frac{ace - bcd}{ae - bd} = c$.
 2dly, $dx = \frac{cde - bdf}{ae - bd}$, and $ey = \frac{aef - cde}{ae - bd}$; therefore $dx + ey = \frac{aef - bdf}{ae - bd} = f$. Q. E. D.

PROBLEM 19.

153. Two persons A and B were talking of their money: says A to B, give me q of your money, and I shall then have r times as much as you will have left; says B to A, give me q of your money, and I shall then have s times as much as you will have left: How much money had each?

SOLUTION.

For A's and B's money put x and y respectively; and the fundamental equations will be $x + q = ry - qr$, and

$$y + q = sx - qs;$$

and these equations when reduced to the form for the last problem, will stand thus;

$$x - ry = -q - qr, \text{ and}$$

$$sx - y = +q + qs.$$

Dispose the coefficients and absolute terms as in the last problem, and they will stand thus;

$$\begin{array}{r} 1 \qquad \qquad \qquad s \\ -q - qr \qquad \qquad +q + qs \\ -r \qquad \qquad \qquad -1 \end{array}$$

Compare the first couple with the second, and you will have $1 \times \overline{q + qs} = q + qs$, and $s \times \overline{-q - qr} = -qs - qrs$; subtract the latter product from the former, and you will have $q + 2qs + qrs$ for the numerator of y ; compare the second couple with the third, and you will have $-q - qr \times -1 = +q + qr$, and $q + qs \times -r = -qr - qrs$; subtract

tract the latter product from the former, and you will have $q + 2qr + qrs$ for the numerator of x ; compare the first couple with the last, and you will have $1x - 1 = -1$, and $sx - r = -rs$; subtract the latter product from the former, and you will have $rs - 1$ for a common denominator; therefore x (or A 's money) was $\frac{q + 2qr + qrs}{rs - 1}$; and y (or B 's money) was $\frac{q + 2qs + qrs}{rs - 1}$.

If q be taken equal to $rs - 1$, it will destroy the denominator, and the solution will come out in whole numbers, thus; A 's money will be $1 + 2r + rs$, and B 's will be $1 + 2s + rs$; as for example, let $r = 3$, and $s = 7$; then you will have $rs - 1 = 20$; put $q = 20$, and the problem will stand thus; says A to B , give me 20 shillings of your money, and I shall then have three times as much as you will have left; says B to A , give me 20 shillings of your money, and I shall then have 7 times as much as you will have left: the answer to which is, that A 's money, or $1 + 2r + rs = 28$ shillings; and B 's money, or $1 + 2s + rs = 36$ shillings.

Note, That if A upon receiving q of B 's money had a sum equal to what B had left, in that case r must be made equal to 1.

THE GENERAL DEMONSTRATION.

1st, If to A 's money, to wit, $\frac{q + 2qr + qrs}{rs - 1}$, be added q , the sum will be $\frac{2qr + 2qrs}{rs - 1}$; and if from B 's money, to wit, $\frac{q + 2qs + qrs}{rs - 1}$, be subtracted q , the remainder will be $\frac{2q + 2qs}{rs - 1}$, which if multiplied by r , will be equal to the former sum; and therefore the first condition of the problem is answered.

2dly, If to B 's money be added q , the sum will be found to be $\frac{2qs + 2qrs}{rs - 1}$; and if from A 's money be subtracted q , the remainder will be found to be $\frac{2q + 2qr}{rs - 1}$, which being multiplied by s , will be equal to the former sum; and therefore the second condition of the problem is answered. *Q. E. D.*

PROBLEM 20. (See art. 81.)

154. It is required to find two numbers x and y of such a nature, that if both be multiplied into 1, the first product shall be a square, and the second

Art. 154, 155. AND THEOREMS DEDUCED FROM THEM. 241

second the side or root of that square; but if both be multiplied into s, the first product shall be a cube, and the second the root of that cube.

SOLUTION.

The numbers x and y being multiplied both into r , produce rx and ry , whereof the former is to be the square of the latter; whence we have this equation, $rx = r^2y^2$, and $x = ry^2$: again, x and y multiplied both into s , produce sx and sy , whereof the former is to be the cube of the latter; whence we have this equation, $sx = s^3y^3$ and $x = s^2y^3$; therefore $s^2y^3 = ry^2$, because both are equal to x ; divide both sides by y^2 , and you will have $s^2y = r$, and $y = \frac{r}{s^2}$; but if $y = \frac{r}{s^2}$, $y^2 = \frac{r^2}{s^4}$, and ry^2 or $x = \frac{r^3}{s^4}$; therefore the numbers are, $x = \frac{r^3}{s^4}$, and $y = \frac{r}{s^2}$.

DEMONSTRATION.

If the numbers $\frac{r^3}{s^4}$ and $\frac{r}{s^2}$ be multiplied both into r , their products will be $\frac{r^4}{s^4}$ and $\frac{r^2}{s^2}$, whereof the former is a square, and the latter is the root of that square; and if the same numbers $\frac{r^3}{s^4}$ and $\frac{r}{s^2}$ be multiplied both into s , the products will be $\frac{r^3}{s^3}$ and $\frac{r}{s}$, whereof the former is a cube, and the latter the root of that cube. Q. E. D.

PROBLEM 21. (See art. 130.)

155. *What two numbers are those, whose difference being multiplied into the difference of their squares will make a, and whose sum being multiplied into the sum of their squares will make b?*

SOLUTION.

For the two numbers sought put x and y ; then according to the first supposition, $x - y \times x^2 - y^2$, or $x - y \times x - y \times x + y$, or $xx - 2xy + y^2 \times x + y = a$; therefore

$$\text{Equ. 1st, } x^2 - 2xy + y^2 = \frac{a}{x+y}.$$

Again, according to the second supposition, $x + y \times x^2 + y^2 = b$; therefore

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Equ. 2d, $x^2 + y^2 = \frac{b}{x+y}$.

From twice the second equation, subtract the first,

that is, from $2x^2 + 2y^2 = \frac{2b}{x+y}$

subtract $x^2 - 2xy + y^2 = \frac{a}{x+y}$

and there will remain $x^2 + 2xy + y^2 = \frac{2b-a}{x+y}$,

that is, $(x+y)^2 = \frac{2b-a}{x+y}$; therefore $(x+y)^3 = 2b-a$; make $2b-a = r^3$, that is, put r for the cube root of $2b-a$, and you will have

Equ. 3d, $x+y=r$.

Again, in the first equation we had $x^2 - 2xy + y^2 = \frac{a}{x+y} = \frac{a}{r}$, that is, $(x-y)^2 = \frac{a}{r}$; make $\frac{a}{r} = s^2$, that is, put s for the square root of $\frac{a}{r}$, and you will have

Equ. 4th, $x-y=s$.

Add the third and fourth equations together, and you will have $2x = r+s$, and $x = \frac{r+s}{2}$; subtract the fourth equation from the third, and

you will have $2y = r-s$, and $y = \frac{r-s}{2}$; whence we have the following canon:

Make $2b-a = r^3$, and $\frac{a}{r} = s^2$, and the numbers sought will be $\frac{r+s}{2}$, and $\frac{r-s}{2}$.

DEMONSTRATION.

The difference of the numbers $\frac{r+s}{2}$ and $\frac{r-s}{2}$ is s , and the difference of their squares is rs , as is easily tried; therefore the difference of the numbers multiplied into the difference of their squares is $rs s = \frac{ra}{r} = a$:

again, the sum of the numbers $\frac{r+s}{2}$ and $\frac{r-s}{2}$ is r , and the sum of their

Art. 155, 156. AND THEOREMS DEDUCED FROM THEM. 243

their squares is $\frac{r^2 + s^2}{2}$; therefore the sum of the numbers multiplied into the sum of their squares is $\frac{r^3 + r s s}{2}$; but $r^3 = 2b - a$ by the canon, and $r s s = a$ by the same; therefore the sum of the numbers multiplied into the sum of their squares is $\frac{2b - a + a}{2} = b$. Q. E. D.

PROBLEM 22.

156. Out of a common pack of fiftytwo cards, let part be distributed into several distinct parcels or heaps in the manner following: upon the lowest card of every heap let as many others be laid as are sufficient to make up it's number twelve; as if four be the number of the lowest card, let eight others be laid upon it; if five, let seven; if a, let twelve — a, &c: It is required, having given the number of heaps, which we shall call n , as also the number of cards still remaining in the dealer's hand, which we shall call r , to find the sum of the numbers of all the bottom cards put together.

SOLUTION.

Let a, b, c , &c express the number of the bottom card in the several heaps; then will $12 - a$ express the number of all the cards lying upon the bottom card of the first heap, that is, the number of all the cards of the first heap except the lowest, will be $12 - a$; therefore $13 - a$ will be the number of all the cards in the first heap; for the same reason, $13 - b$ will be the number of all the cards in the second heap; and $13 - c$ the number of all those in the third, and so on; therefore the number of all the cards in all the heaps will be $13 \times n - a - b - c$ &c: make $a + b + c$ &c (or the sum of the numbers of all the bottom cards) $= x$, and then we shall have the number of all the cards drawn out into heaps $= 13n - x$; but these, together with r , the number of cards undrawn out, make up the whole pack 52; therefore we have this equation, $13n - x + r = 52$; therefore $x + 52 = 13n + r$; therefore $x = 13n - 52 + r$; but $52 = 13 \times 4$; therefore $13n - 52 = 13 \times n - 4$; therefore $x = 13 \times n - 4 + r$; in words thus: From the number of heaps subtract four; multiply the rest by thirteen; and this product added to the number of cards still remaining in the dealer's hand, will give the sum of the numbers of all the bottom cards put together: as for example, let there be three heaps, and thirty cards remaining; now 4 subtracted from 3 leaves —1; this multiplied by 13 gives —13, and this product added

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to 30, the number of cards remaining, gives 17 for the sum of the numbers of all the bottom cards.

A more universal theorem is as follows :

Let n be the number of heaps as before, p the number of cards in a pack ; let as many cards be laid upon the lowest of every heap as are sufficient to make up it's number q ; and lastly, let r be the number of remaining cards as before ; and the sum of the numbers of all the bottom cards will be found to be $q + 1 \times n + r - p$.

PROBLEM 23.

157. It is required to determine the values of three unknown quantities x , y and z by the help of three equations of the following form, to wit, $px + qy + rz = s$, where the quantities represented by p , q , r , s , are all supposed to be known.

N. B. I know not whether I have not already been too tedious in my 18th problem, and therefore I shall not trouble my reader with the investigation of the following theorem ; I shall only give him a few hints, which, if at any time he attempts it himself, may be of some use to him : but first I shall give him the theorem itself, which is as follows :

SOLUTION.

First put down the three coefficients of x one after another as they stand in the three equations, which coefficients we will call a , b , c ; under these, put down the three absolute terms, suppose d , e , f ; under these again write down the coefficients of z , suppose g , h , k ; and under these write the coefficients of y , suppose l , m , n : from these four serieses thus put down, must be derived two others, to wit, α , β , γ , and δ , ϵ , ζ ; and from these two last a third, to wit, η , θ , κ , all which must be written down as in the following diagram † ; and the terms of the three last serieses must be had by a cross multiplication thus : make $ae - bd = \alpha$, $ah - bg = \beta$, $am - bl = \gamma$; make also $bf - ce = \delta$, $bk - ch = \epsilon$, and $bn - cm = \zeta$; lastly, make $\alpha\epsilon - \delta\beta = \eta$, $\alpha\zeta - \delta\gamma = \theta$, and $\beta\zeta - \epsilon\gamma = \kappa$, and this last series η , θ , κ will solve the problem : for y will be $\frac{-\eta}{\kappa}$, and z will be $\frac{+\theta}{\kappa}$; whence x will easily be had by any of the three fundamental equations, if instead of y and z be substituted their values here found.

†		
a	b	c
d	e	f
g	h	k
l	m	n
α		δ
β		ϵ
γ		ζ
	η	
	θ	
	κ	

Who-

Whoever attempts the investigation of this theorem, had best do it by the method laid down in the 18th problem; for if $px + qy + rz = s$, we shall have $px + qy = s - rz$; where the quantity $s - rz$ must be looked upon as the absolute term: this observed, if he makes use of the coefficients contained in the serieses above described, and applies the method of the 18th problem to the two first equations, he will find the value of y ,

when contracted by our notation, to be $\frac{\alpha - \beta z}{\gamma}$; and if he applies the

same method to the second and third equations, he will find $y = \frac{\delta - \epsilon z}{\zeta}$;

therefore he will have this equation, $\frac{\alpha - \beta z}{\gamma} = \frac{\delta - \epsilon z}{\zeta}$; which being

resolved and contracted, gives $z = \frac{\theta}{\kappa}$: and when he considers how the terms $\alpha, \beta, \gamma, \&c$ were obtained, he will naturally fall into the method above described; and lastly, when he has got this theorem for determining the value of z , he will as easily find the value of y , to wit, by putting the coefficients of y in the place of those of z , and *vice versa*.

N. B. Coming now to give some examples of the solution of general problems producing quadratic equations, I must advertise the reader once for all, that whenever I am to express the square root of any quantity, which properly speaking, is no square, to avoid the introducing of surd roots, I do it for the most part by the letter s , putting s^2 for the number whose square root is to be signified by s : thus if I am to express the

square root of $\frac{aa - 4b}{4}$, because the denominator 4 is a square number,

I exclude it out of the value of s^2 , and so put s^2 for $aa - 4b$, whence

$\frac{aa - 4b}{4} = \frac{s^2}{4}$, and it's root will be $\frac{s}{2}$; but if I was to express the square

root of $\frac{4b - a^3}{12a}$, since here neither the numerator nor denominator are

square numbers, I might put ss for the whole fraction; but consider-

ing that $\frac{4b - a^3}{12a}$ is the product of $\frac{1}{4}$ multiplied into $\frac{4b - a^3}{3a}$, whereof

the former factor is a square, and the latter is no square, I rather choose

to put ss for the latter factor $\frac{4b - a^3}{3a}$, and so make $\frac{4b - a^3}{12a} = \frac{ss}{4}$.

Note also, That I shall resolve all the following equations in the ordinary way, without having recourse to the general theorem in art. 103.

PROBLEM 24. (See art. III.)

158. What two numbers are those, whose sum is a , and the product of whose multiplication is b ?

SOLUTION.

The two numbers sought, x and $a-x$.

The product of their multiplication, $ax - xx = b$; whence changing the signs, $xx - ax = -b$, and completing the square, $xx - ax + \frac{aa}{4} = \frac{aa}{4} - b = \frac{aa - 4b}{4} = \frac{ss}{4}$; extract the square root of both sides, that is, of $xx - ax + \frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x - \frac{a}{2} = \pm \frac{s}{2}$, and $x = \frac{a \pm s}{2}$; whence the following canon:

Make $aa - 4b = ss$, and the greater number will be $\frac{a+s}{2}$, and the less number $\frac{a-s}{2}$.

The SYNTHETICAL DEMONSTRATION.

1st, $\frac{a+s}{2}$ added to $\frac{a-s}{2}$ gives $\frac{2a}{2}$ or a .

2dly, $\frac{a+s}{2}$ multiplied into $\frac{a-s}{2}$ gives $\frac{aa - ss}{4} =$ (by substituting $-aa + 4b$ instead of $-ss$) $\frac{aa - aa + 4b}{4} = \frac{4b}{4} = b$. Q. E. D.

An example to the foregoing canon.

What two numbers are those, whose sum is twentyfive, and the product of whose multiplication is 144? Here $a = 25$, $b = 144$, $aa - 4b$ or $ss = 49$, $s = 7$, $\frac{a+s}{2} = 16$, $\frac{a-s}{2} = 9$; so the numbers are 9 and 16.

PROBLEM 25. (See art. III.)

159. What two numbers are those, whose sum is a , and the sum of their squares b ?

SOLUTION.

SOLUTION.

The two numbers fought, x and $a - x$.

The square of the former, xx .

The square of the latter, $aa - 2ax + xx$.

The sum of their squares $aa - 2ax + 2xx = b$; therefore $2xx - 2ax = b - aa$, and $xx - ax = \frac{b - aa}{2}$, and $xx - ax + \frac{aa}{4} = \frac{aa}{4} + \frac{b - aa}{2} = \frac{2b - aa}{4} = \frac{ss}{4}$; extract the square roots, that is, the root of $xx - ax + \frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x - \frac{a}{2} = \pm \frac{s}{2}$, and $x = \frac{a \pm s}{2}$; whence the following canon:

Make $2b - aa = ss$, and you will have $\frac{a + s}{2}$ for the greater number, and $\frac{a - s}{2}$ for the less.

DEMONSTRATION.

1st, $\frac{a + s}{2}$ added to $\frac{a - s}{2}$ gives a .

2dly, The square of $\frac{a + s}{2}$ is $\frac{aa + 2as + ss}{4}$; the square of $\frac{a - s}{2}$ is $\frac{aa - 2as + ss}{4}$; and therefore the sum of their squares is $\frac{2aa + 2ss}{4} = \frac{aa + ss}{2} =$ (by the canon) $\frac{aa + 2b - aa}{2} = b$. Q. E. D.

An example to the foregoing canon.

What two numbers are those, whose sum is 28, and the sum of their squares 400? Here $a = 28$, $b = 400$, $2b - aa$ or $ss = 16$, $s = 4$, $\frac{a + s}{2} = 16$, $\frac{a - s}{2} = 12$; therefore the numbers are 12 and 16.

PROBLEM 26. (See art. 114.)

160. What two numbers are those, whose sum is a , and the sum of their cubes b ?

SOLU-

SOLUTION.

The two numbers sought, x and $a - x$.

The cube of the former, x^3 .

The cube of the latter, $a^3 - 3a^2x + 3ax^2 - x^3$.

The sum of their cubes, $a^3 - 3a^2x + 3ax^2 = b$; therefore $3ax^2 - 3a^2x = b - a^3$; divide by $3a$, and you will have $xx - ax = \frac{b - a^3}{3a}$, and $xx - ax + \frac{aa}{4} = \frac{aa}{4} + \frac{b - a^3}{3a} = \frac{4b - a^3}{12a} = \frac{1}{4} \times \frac{4b - a^3}{3a} = \frac{ss}{4}$; extract the square root of both sides, that is, of $xx - ax + \frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x - \frac{a}{2} = \pm \frac{s}{2}$ and $x = \frac{a \pm s}{2}$; whence the following canon:

Make $\frac{4b - a^3}{3a} = ss$, and you will have $\frac{a + s}{2}$ for the greater number, and $\frac{a - s}{2}$ for the less.

DEMONSTRATION.

1st, $\frac{a + s}{2}$ added to $\frac{a - s}{2}$ gives a .

2dly, The cube of $\frac{a + s}{2}$ is $\frac{a^3 + 3a^2s + 3as^2 + s^3}{8}$, and the cube of $\frac{a - s}{2}$ is $\frac{a^3 - 3a^2s + 3as^2 - s^3}{8}$; therefore the sum of their cubes is $\frac{2a^3 + 6as^2}{8} = \frac{a^3 + 3as^2}{4} = \frac{a^3 + 4b - a^3}{4}$ by the canon, $= b$. Q. E. D.

An example to the foregoing canon.

What two numbers are those, whose sum is 7, and the sum of their cubes 133? Here $a = 7$, $b = 133$, $\frac{4b - a^3}{3a}$ or $ss = 9$, $s = 3$, $\frac{a + s}{2} = 5$, $\frac{a - s}{2} = 2$; so the numbers are 5 and 2.

PROBLEM 27.

161. It is required to find two numbers whose difference is d , and which dividing a given number as a , will have two quotients whose difference is b .

SOLU-

SOLUTION.

The two numbers sought, x and $x + d$.

The two quotients, $\frac{a}{x}$ and $\frac{a}{x+d}$.

Their difference, $\frac{a}{x} - \frac{a}{x+d} = \frac{ad}{xx+dx} = b$; therefore $bx + bdx = ad$, and $xx + dx = \frac{ad}{b}$; therefore $xx + dx + \frac{dd}{4} = \frac{ad}{b} + \frac{dd}{4} = \frac{1}{4}x^2 + \frac{dd}{4} = \frac{ss}{4}$; extract the square root of $xx + dx + \frac{dd}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x + \frac{d}{2} = \pm \frac{s}{2}$, whence $x = \frac{s-d}{2}$ or $\frac{-s-d}{2}$; set aside the negative root, and you will have x (the less divisor) $= \frac{s-d}{2}$, and $x + d$ (the greater) $= \frac{s-d}{2} + \frac{d}{1} = \frac{s+d}{2}$; and we shall have the following canon:

Make $\frac{4ad}{b} + dd = ss$, and you will have $\frac{s+d}{2}$ for the greater divisor, and $\frac{s-d}{2}$ for the less.

N. B. That $\frac{s-d}{2}$ is an affirmative quantity, is evident from hence, that $ss = \frac{4ad}{b} + dd$; therefore ss is greater than dd , and s greater than d ; therefore $\frac{s-d}{2}$ is affirmative.

The demonstration of the canon.

1st, If the less divisor $\frac{s-d}{2}$ be subtracted from the greater $\frac{s+d}{2}$, the remainder will be d ; therefore the difference of the divisors is d .

2dly, If the dividend a be severally divided by the two divisors $\frac{s-d}{2}$ and $\frac{s+d}{2}$, the two quotients will be $\frac{2a}{s-d}$ and $\frac{2a}{s+d}$ respectively, whereof the former will be the greater, as having a less denominator; therefore

fore the difference of the quotients is $\frac{2a}{s-d} - \frac{2a}{s+d} = \frac{2as+2ad-2as+2ad}{ss-dd} = \frac{4ad}{ss-dd} = \frac{4ad}{\frac{4ad}{b}}$ by the canon, $=b$. Q. E. D.

An example to the foregoing canon.

Let it be required to find two divisors whose difference is 1, and which dividing a given number as 144, will have two quotients whose difference is 2. Here $a=144$, $b=2$, $d=1$, $\frac{4ad}{b} + dd$ or $ss=289$, $s=17$, $\frac{s+d}{2}=9$, $\frac{s-d}{2}=8$; therefore the divisors are 8 and 9, and the quotients 18 and 16.

SCHOLIUM.

If in this last problem we had put x for the greater quantity, and $x-d$ for the less, the equation would have been $\frac{a}{x-d} - \frac{a}{x} = b$, or $\frac{ad}{xx-dx} = b$, which is different from the former; and therefore it could not be expected, that in that equation, the two roots should be the numbers sought, but rather the two different values of x , the lesser of them.

PROBLEM 28. (See art. 118.)

162. *What number is that, which being added to it's square root, will make a?*

SOLUTION.

Put xx for the number sought, and you will have this equation, $xx + 1x = a$; therefore $xx + 1x + \frac{1}{4} = a + \frac{1}{4} = \frac{4a+1}{4} = \frac{ss}{4}$; therefore $x + \frac{1}{2} = \pm \frac{s}{2}$; therefore $x = \frac{s-1}{2}$ or $\frac{-s-1}{2}$: If x be made $= \frac{s-1}{2}$, you will have $xx = \frac{ss-2s+1}{4}$; if x be made equal to $\frac{-s-1}{2}$, you will have $xx = \frac{ss+2s+1}{4}$; whence the following canon:

Make $4a+1=ss$, and the number sought will be $\frac{ss-2s+1}{4}$ or $\frac{ss+2s+1}{4}$.

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$\frac{ss+2s+1}{4}$, according as the square root to be added is taken affirmatively or negatively.

DEMONSTRATION.

Case 1st, If to the number $\frac{ss-2s+1}{4}$ be added it's affirmative square root $\frac{s-1}{2}$, or $\frac{2s-2}{4}$, the sum will be $\frac{ss-1}{4} = a$, by the canon.

Case 2d, If to the number $\frac{ss+2s+1}{4}$ be added it's negative square root $\frac{-s-1}{2}$ or $\frac{-2s-2}{4}$, the sum will again be $\frac{ss-1}{4} = a$, as before.

Q. E. D.

PROBLEM 29. (See art. 129.)

163. It is required to find three numbers in continual proportion, whose sum is a , and the sum of their squares ab .

Therefore b (or $\frac{ab}{a}$) is the quotient of the sum of the squares divided by the sum of the numbers; which way of notation will be of some small advantage in the computation that follows.

SOLUTION.

For the three numbers sought put x , y and z ; then since these numbers are in continual proportion, that is, since x is to y as y is to z , we have $yy = xz$; again, since by the supposition, $x + y + z = a$, we have $a - y = x + z$, and squaring both sides, $a^2 - 2ay + yy = xx + 2xz + zz$; subtract y^2 from one side, and it's equal xz from the other, and you will have $aa - 2ay = xx + xz + z^2 = x^2 + y^2 + z^2 = ab$; since then $aa - 2ay = ab$, divide by a and you will have $a - 2y = b$, and y (the middle term) $= \frac{a-b}{2}$; subtract this from a , the sum of all the three

numbers sought, and you will have $\frac{a+b}{2}$ for the sum of the extremes;

let us call this sum $2l$, and let us also call $\frac{a-b}{2}$ or the middle term, m ; then as x is one of the extremes, $2l - x$ will be the other; and the product of the extremes will be $2lx - xx = m^2$; therefore $xx - 2lx = -m^2$, and $xx - 2lx + l^2 = l^2 - m^2 = n^2$, and $x - l = \pm n$, and $x = l \pm n$; whence may be derived the following canon:

I i 2

Make

Make $\frac{a+b}{2} = 2l$, $\frac{a-b}{2} = m$, and $l^2 - m^2 = n^2$, and the three numbers sought will be $\overline{l+n}$, m and $\overline{l-n}$.

DEMONSTRATION.

1st, Since by the supposition $n^2 = l^2 - m^2$, we have $m^2 = l^2 - n^2 = \overline{l+n} \times \overline{l-n}$; therefore the three numbers $\overline{l+n}$, m and $\overline{l-n}$ are in continual proportion, by the latter end of the 15th article, because the square of the middle term is equal to the product of the extremes; which answers the first condition of the problem.

2dly, If the three numbers $\overline{l+n}$, m , and $\overline{l-n}$ be added together, their sum will be $2l + m = \frac{a+b}{2} + \frac{a-b}{2}$ by the canon, $= a$; which answers the second condition: in like manner, if m or $\frac{a-b}{2}$ be subtracted from $2l$ or $\frac{a+b}{2}$, you will have $2l - m = b$.

3dly, The square of $\overline{l+n}$ is $l^2 + 2ln + n^2$, the square of m is m^2 , and the square of $\overline{l-n}$ is $l^2 - 2ln + n^2$; add these three squares together, and their sum will be $2l^2 + m^2 + 2n^2 = 2l^2 + m^2 + 2l^2 - 2m^2$ by the canon, $= 4l^2 - m^2 = 2\overline{l+m} \times 2\overline{l-m} = ab$, by the second step; therefore all the conditions of the problem are satisfied. Q. E. D.

An example to the foregoing canon.

Let the sum of the three numbers be 19, and let the sum of their squares be 133; then we shall have $a = 19$, $b = \frac{133}{19} = 7$, $\frac{a+b}{2}$ or $2l = 13$, $l = \frac{13}{2}$, $\frac{a-b}{2}$ or $m = 6$, $l^2 - m^2$ or $n^2 = \frac{25}{4}$, $n = \frac{5}{2}$, $l+n = 9$, $l-n = 4$; therefore the numbers are 9, 6 and 4, or 4, 6 and 9.

PROBLEM 30.

164. It is required to find four numbers in continual proportion, and such, that the sum of the extremes may be a , and that of the middle terms b .

Note, For what is meant by 4 numbers being in continual proportion, see art. 128.

SOLU-

SOLUTION.

For the two middle terms put x and y ; then the extremes will be found, one by saying, as y is to x so is x to $\frac{xx}{y}$, and the other by saying, as x is to y so is y to $\frac{yy}{x}$; so the extremes are $\frac{xx}{y}$ and $\frac{yy}{x}$, and their sum $\frac{x^2}{y} + \frac{yy}{x} = \frac{x^3 + y^3}{xy}$; therefore the fundamental equations are, 1st, $x + y = b$, or $x = b - y$; and 2^{dly}, $\frac{x^3 + y^3}{xy} = a$, or $x^3 + y^3 = axy$; instead of x in this last equation use $b - y$, it's value in the first, and you will have $x^3 = b^3 - 3bby + 3byy - y^3$, and $x^3 + y^3 = b^3 - 3bby + 3byy$; you will also have axy , or $ay \times b - y = aby - ayy$; therefore the equation will now be $3byy - 3bby + b^3 = aby - ayy$, transpose b^3 , as also $aby - ayy$, and then the equation will stand thus, $ayy + 3byy - aby - 3bby = -b^3$, or thus, $yy - by \times a + 3b = -b^3$; therefore $yy - by = \frac{-b^3}{a + 3b}$, and $yy - by + \frac{bb}{4} = \frac{bb}{4} - \frac{b^3}{a + 3b} = \frac{abb - b^3}{4a + 12b} = \frac{bb}{4} \times \frac{a - b}{a + 3b} = \frac{bb}{4} \times ss = \frac{bbss}{4}$; extract the square root of $yy - by + \frac{bb}{4}$ on one side, and of $\frac{bbss}{4}$ on the other, and you will have $y - \frac{b}{2} = \pm \frac{bs}{2}$, and $y = \frac{b}{2} \pm \frac{bs}{2} = \frac{b}{2} \times 1 \pm s$; and since the equation will be the same, which soever of the two middle terms y is made to stand for, the greater or the less, we have the following canon:

Make $\frac{a - b}{a + 3b} = ss$, and you will have $\frac{b}{2} \times 1 + s$ for the greater of the two middle terms, and $\frac{b}{2} \times 1 - s$ for the less.

The middle terms then being thus obtained by the help of the foregoing canon, the extremes will easily be had from the nature of continual proportionality as above, to wit, by saying, As $\frac{b}{2} \times 1 - s$ is to $\frac{b}{2} \times 1 + s$, or as $1 - s$ is to $1 + s$, so is $\frac{b}{2} \times 1 + s$ to $\frac{b}{2} \times \frac{1 + s}{1 - s}$; therefore $\frac{b}{2} \times \frac{1 + s}{1 - s}$ is

is the greater extreme: and as $\frac{b}{2} \times \frac{1+s}{1-s}$ is to $\frac{b}{2} \times \frac{1-s}{1+s}$, or as $1+s$ is to $1-s$, so is $\frac{b}{2} \times \frac{1-s}{1+s}$ to $\frac{b}{2} \times \frac{1+s}{1-s}$; therefore $\frac{b}{2} \times \frac{1-s}{1+s}$ is the lesser extreme; and the four numbers, putting the greater extreme first, will stand thus; $\frac{b}{2} \times \frac{1+s}{1-s}$, $\frac{b}{2} \times \frac{1-s}{1+s}$, $\frac{b}{2} \times \frac{1-s}{1+s}$ and $\frac{b}{2} \times \frac{1+s}{1-s}$.

DEMONSTRATION.

That these four numbers must be continual proportionals is plain; for the extremes were found upon that supposition: and that the two middle terms when added together must make b is as plain; for $\frac{b}{2} \times \frac{1+s}{1-s} + \frac{b}{2} \times \frac{1-s}{1+s} = \frac{b}{2} \times 2 = b$; therefore the main business of this demonstration will be to prove that the sum of the extremes is a ; we are there-

fore to enquire whether $\frac{b}{2} \times \frac{1+s}{1-s} + \frac{b}{2} \times \frac{1-s}{1+s} = a$, or dividing by $\frac{b}{2}$,

whether $\frac{1+s}{1-s} + \frac{1-s}{1+s} = \frac{2a}{b}$: now $\frac{1+s}{1-s}$ and $\frac{1-s}{1+s}$ being added toge-

ther make $\frac{1+s+1-s}{1-ss}$; but $1+s = 1+3s+3ss+s^3$, and $1-s =$

$1-3s+3ss-s^3$; therefore $\frac{1+s+1-s}{1-ss} = \frac{2+6ss}{1-ss}$; therefore

the enquiry is now reduced to this, whether $\frac{2+6ss}{1-ss} = \frac{2a}{b}$: now to determine this, both the numerator $2+6ss$, and the denominator $1-ss$,

must be examined apart thus; $ss = \frac{a-b}{a+3b}$ by the canon; therefore

$6ss = \frac{6a-6b}{a+3b}$; therefore $2+6ss$ or the numerator $= \frac{2}{1} + \frac{6a-6b}{a+3b}$

$= \frac{8a}{a+3b}$: again, $ss = \frac{a-b}{a+3b}$; therefore $1-ss = \frac{-a+b}{a+3b}$; therefore

$1-ss,$

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$1 - ss$, or the denominator, equals $\frac{1 - a + b}{1 - a + 3b} = \frac{4b}{a + 3b}$; therefore $\frac{2 + 6ss}{1 - ss}$ is equal to a fraction whose numerator is $\frac{8a}{a + 3b}$, and whose denominator is $\frac{4b}{a + 3b}$; but such a fraction is equal to $\frac{8a}{4b}$ or $\frac{2a}{b}$; therefore $\frac{2 + 6ss}{1 - ss}$, or the sum of the extremes after being divided by $\frac{b}{2}$, equals $\frac{2a}{b}$; therefore before any such division was made, the sum of the extremes was a . Q. E. D.

An example to the foregoing canon.

Let the sum of the extremes be 27, that of the middle terms 18; and you will have $a = 27$, $b = 18$, $\frac{a - b}{a + 3b}$ or $ss = \frac{9}{81} = \frac{1}{9}$, $s = \frac{1}{3}$, $1 + s = \frac{4}{3}$, $1 - s = \frac{2}{3}$, $\frac{b}{2} \times 1 + s = 12$, $\frac{b}{2} \times 1 - s = 6$; and so the two middle terms are 12 and 6; whence the extremes are 24 and 3; and the 4 numbers in order are 24, 12, 6 and 3, or 3, 6, 12 and 24.

PROBLEM. 31.

165. *What two numbers are those, whose sum added to the sum of their squares is a, and whose difference added to the difference of their squares is b?*

SOLUTION.

Put x and y for the two numbers sought, and the fundamental equations will be 1st, $x + y + x^2 + y^2 = a$; 2dly, $x - y + x^2 - y^2 = b$; which equations when reduced to order will stand thus;

Equ. 1st, $xx + x + yy + y = a$.

Equ. 2d, $xx + x - yy - y = b$.

Add these two last equations together, and you will have $2xx + 2x = a + b$;

whence $xx + 1x = \frac{a + b}{2}$, and $xx + 1x + \frac{1}{4} = \frac{a + b}{2} + \frac{1}{4} = \frac{2a + 2b + 1}{4}$
 $= \frac{rr}{4}$; extract the root of $xx + 1x + \frac{1}{4}$ on one side, and of $\frac{rr}{4}$ on the

other, and you will have $x + \frac{1}{2} = \frac{r}{2}$, and $x = \frac{r - 1}{2}$: again, subtract the second equation from the first, and you will have $2y^2 + 2y = a - b$;

$-b$; and $y^2 + y = \frac{a-b}{2}$, and $y^2 + 1y + \frac{1}{4} = \frac{2a-2b+1}{4} = \frac{ss}{4}$;

whence $y + \frac{1}{2} = \frac{s}{2}$, and $y = \frac{s-1}{2}$; whence the following canon:

Make $2a+2b+1=rr$, and $2a-2b+1=ss$, and you will have the greater number equal to $\frac{r-1}{2}$, and the less number $=\frac{s-1}{2}$.

DEMONSTRATION.

The sum of $\frac{r-1}{2}$ and $\frac{s-1}{2}$ is $\frac{r+s-2}{2}$ or $\frac{2r+2s-4}{4}$.

The square of $\frac{r-1}{2}$ is $\frac{r^2-2r+1}{4}$.

The square of $\frac{s-1}{2}$ is $\frac{s^2-2s+1}{4}$.

Therefore the sum of their squares is $\frac{r^2+s^2-2r-2s+2}{4}$; add to

this the sum of the numbers above found, to wit, $\frac{2r+2s-4}{4}$, and you

will have the sum of the numbers added to the sum of their squares equal to $\frac{r^2+ss-2}{4}$; but $r^2+s^2=4a+2$ by the canon; therefore

$rr+ss-2=4a$, and $\frac{r^2+s^2-2}{4}$, or the sum of the numbers added

to the sum of their squares, equals a : again, the difference of $\frac{r-1}{2}$

and $\frac{s-1}{2}$ is $\frac{r-s}{2}$ or $\frac{2r-2s}{4}$; and the difference of their squares is

$\frac{r^2-s^2+2s-2r}{4}$; therefore the difference of the numbers added to the

difference of their squares is $\frac{r^2-s^2}{4} = \frac{4b}{4}$ by the canon, $=b$. Q. E. D.

An example to the foregoing canon.

Let the sum of the numbers added to the sum of their squares be 26, and their difference added to the difference of their squares 14; and we

shall have $a=26$, $b=14$, $2a+2b+1$ or $rr=81$, $r=9$, $\frac{r-1}{2}$

$=4$, $2a-2b+1$ or $ss=25$, $s=5$, $\frac{s-1}{2}=2$; and so the numbers

sought will be 4 and 2.

P R O-

PROBLEM 32.

166. What two numbers are those, the sum of whose squares is a , and the product of their multiplication b ?

SOLUTION.

For the two numbers sought put x and $\frac{b}{x}$, and the sum of their squares will be $x^2 + \frac{b^2}{x^2} = a$; therefore $x^4 + b^2 = ax^2$; therefore $x^4 - ax^2 = -bb$, and $x^4 - ax^2 + \frac{aa}{4} = \frac{aa}{4} - bb = \frac{aa - 4bb}{4} = \frac{ss}{4}$; extract the square root of $x^4 - ax^2 + \frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x^2 - \frac{a}{2} = \pm \frac{s}{2}$, and $x^2 = \frac{a \pm s}{2}$; and since this equation will be the same, whichever of the unknown quantities x is made to stand for, you will have the following canon:

Make $aa - 4bb = ss$, and you will have the square of the greater number equal to $\frac{a+s}{2}$, and the square of the less equal to $\frac{a-s}{2}$.

DEMONSTRATION.

If the square of the greater number, which is $\frac{a+s}{2}$, be added to the square of the less number, which is $\frac{a-s}{2}$, the sum of their squares will be $\frac{2a}{2}$ or a : again, if the square of the greater number which is $\frac{a+s}{2}$ be multiplied into the square of the less number which is $\frac{a-s}{2}$, the product of these two squares will be $\frac{aa - ss}{4} = \frac{aa - aa + 4bb}{4}$ by the canon, $= \frac{4bb}{4} = bb$; but if the square of the greater number multiplied into the square of the less gives bb , then the greater number multiplied into the less will give b . Q. E. D.

An example to the foregoing canon.

Let the sum of the squares of the two numbers sought be 400, and the product of their multiplication 192; then you will have $a=400$, $b=192$, $a^2 - 4b^2$ or $s^2 = 12544$, $s = 112$, $\frac{a+s}{2}$ or the square of the greater number $= 256$, $\frac{a-s}{2}$ or the square of the less number $= 144$; therefore the greater number is 16, and the less 12.

PROBLEM 33.

167. *Let a stone be dropt into an empty pit or well: It is required, having given the time from the first dropping of the stone to the hearing of the sound from the bottom, to assign the depth of the well.*

SOLUTION.

The solution of this problem is founded upon three philosophical principles, all which must here be taken for granted: The first is, that the spaces through which a heavy body falls from rest in different times, are as the squares of those times: thus the space described in two seconds of time, is to the space described in three seconds, not as 2 to 3, but as 4 to 9. The second is, that setting aside the resistance of the air, which is not here considered, all bodies both great and small, heavy and light, are accelerated alike, and descend through the same space in the same time. The third is, that all sounds, at least, all such as are strong enough to affect our senses, are propagated through the air with the same velocity, and consequently describe spaces proportionable to the time of their motion. The first of these principles was demonstrated long since by it's first discoverer *Galileo*, and the two last by the celebrated Sir *Izaak Newton*. These things then being supposed, let x be put for the depth of the well, let a represent the space through which a heavy body falls from rest in one second of time, b the space through which a sound passes in the same time, and let c be the time given in seconds from the first descent of the stone to the hearing of the sound; then to find how long the stone must be in descending to the bottom of the well, say, as the space a is to the space x , to wit, the whole depth of the well, so is 1, the square of the time wherein the former space is described, to wit, the square of one second, to $\frac{x}{a}$, the square of the time wherein the lat-

ter

ter space is described; therefore $\frac{\sqrt{x}}{\sqrt{a}}$ will represent the time the stone takes in descending through the whole depth of the well: again, to find the time the sound takes in ascending through the same space, say, as the space b is to the space x , so is one second, the time wherein the former space is described, to $\frac{x}{b}$ the time wherein the latter space is described: but these times together, that is, the time of the stone's descent, and the time of the sound's ascent, make up the whole time c ; therefore we have this equation, $\frac{x}{b} + \frac{\sqrt{x}}{\sqrt{a}} = c$; therefore $x + \frac{b\sqrt{x}}{\sqrt{a}} = bc$. This is a sort of a quadratic equation, since x is the square of \sqrt{x} ; or at least it is in the form of a quadratic equation, and must be resolved accordingly. Here then the coefficient of the second term is $\frac{b}{\sqrt{a}}$, whose half is $\frac{b}{2\sqrt{a}}$; the square of this is $\frac{bb}{4a}$, which being added to both sides of the equation, gives $x + \frac{b\sqrt{x}}{\sqrt{a}} + \frac{bb}{4a} = \frac{bb}{4a} + bc = \frac{bb+4abc}{4a}$; make $bb+4abc=ss$, and you will have $x + \frac{b\sqrt{x}}{\sqrt{a}} + \frac{bb}{4a} = \frac{ss}{4a}$; whence extracting the root of both sides, you will have $\sqrt{x} + \frac{b}{2\sqrt{a}} = \pm \frac{s}{2\sqrt{a}}$; therefore $\sqrt{x} = \frac{-b \pm s}{2\sqrt{a}}$, that is, $\sqrt{x} = \frac{-b-s}{2\sqrt{a}}$, or $\frac{-b+s}{2\sqrt{a}}$, whereof the former root is negative, and the latter affirmative; but the nature of this problem will not admit that \sqrt{x} be supposed negative; for we must be consistent with ourselves, and if we suppose \sqrt{x} to be negative in any one part of this solution, it must be so in all the rest; whence it will follow, that $\frac{\sqrt{x}}{\sqrt{a}}$, the time of the descent of the stone through the depth of the well is negative, which is absurd; it is for this reason that I reject the negative root $\frac{-s-b}{2\sqrt{a}}$, and retain the affirmative one $\frac{s-b}{2\sqrt{a}}$, making $\sqrt{x} = \frac{s-b}{2\sqrt{a}}$, and consequently x the depth of the well $= \frac{s-b}{4a}$.

If any one asks, how I came to know that even the accepted root $\frac{s-b}{2\sqrt{a}}$ was affirmative; this was easily discovered by observing, that ss being equal to $bb+4abc$, will be greater than bb ; therefore s will be greater than b , and consequently $s-b$ will be an affirmative quantity, as must also $\frac{s-b}{2\sqrt{a}}$: this might also have been discovered by turning over

all the parts of the equation to one side, and so making $\frac{x}{b} + \sqrt{\frac{x}{a}} - c = 0$; see art. 109; but to proceed: since $x = \frac{s-b}{4a}$, we shall have

$\sqrt{x} = \frac{s-b}{2\sqrt{a}}$, and $\frac{\sqrt{x}}{\sqrt{a}}$, or the time of the descent of the stone to the bottom of the well, equal to $\frac{s-b}{2a}$; we shall also have $\frac{x}{b}$, which was

the time of the found's ascent through the same space, equal to $\frac{s-b}{4ab}$; whence is derived the following canon:

Make $bb+4abc$ equal to ss , and you will have the depth of the well equal to $\frac{s-b}{4a}$, the time of the stone's descent equal to $\frac{s-b}{2a}$, and the time of the found's ascent equal to $\frac{s-b}{4ab}$.

DEMONSTRATION.

If the depth of the well be rightly assigned, the time of the stone's descent, and the time of the found's ascent added together, ought to make up the whole time c , from the dropping of the stone to the hearing of the found: now the time of the stone's descent was $\frac{s-b}{2a}$, or $\frac{2bs-2bb}{4ab}$, multiplying both the numerator and denominator by $2b$; and the time of the

found's ascent was $\frac{s-b}{4ab}$, or $\frac{s^2-2bs+bb}{4ab}$; add these two times together, to wit, $\frac{s^2-2bs+bb}{4ab}$, and $\frac{2bs-2bb}{4ab}$, and the sum will be $\frac{s^2-b^2}{4ab} = \frac{4abc}{4ab}$ by the canon, $= c$. Q. E. D.

SCHO-

S C H O L I U M.

Monsieur *Huygens*, justly celebrated for having been the author of so many curious and useful discoveries, both to the learned world in particular, and to mankind in general, having observed that the length of a pendulum that swings seconds is 3 feet 8 lines and a half, *Paris* measure; and having also by an uncommon sagacity found out this curious theorem, that the time of the descent of a heavy body through half the length of any pendulum, is to the time of one single oscillation of that pendulum either forwards or backwards, as the diameter of a circle is to the circumference; from both these together, and from *Galileo's* theorem above mentioned, has demonstrated, that the space through which a heavy body falls from rest in one second of time, is $15\frac{1}{12}$ *Paris* feet, or more accurately, 15.0957 *Paris* feet, that is, 16.1222 English feet, increasing the number of *Paris* feet in the proportion of 1000 to 1068. This discovery, I say, is owing to the sagacity of that excellent Mathematician Monsieur *Huygens*; nor do we owe less to the care and diligence of our learned countryman Mr. *Derham*, who in his experiments and observations upon the motion of sounds, from a far greater distance than any before him had ever observed them, found it to be at the rate of about 1142 feet in a second: therefore 16.1222 feet is the space which in the resolution of the last problem we called a , and 1142 feet is what we called b ; therefore $4a = 64.4888$, and $4ab = 73646$, and $bb = 1304164$. Having then got b^2 and $4ab$ ready calculated to our hand, which in all cases will be the same, the solution of the foregoing problem will become much easier: as for example, suppose the time from the beginning of the fall to the hearing of the sound was 10 seconds; then we shall have $c = 10$, $4abc = 736460$, $bb + 4abc$ or $ss = 2040624$, $s = 1428.5$, $s - b = 286.5$, $\frac{s-b}{2}$ or 142.25 ; this therefore divided by $4a$ or 64.49, gives 1273 feet for the depth of the well: moreover the time of the stone's descent was $\frac{s-b}{2a}$ by the canon; but $s - b$ is found already to be 286.5; and this divided by $2a$ or 32.24, gives 8.89 seconds for the time of the stone's descent: lastly, the time of the sound's ascent was $\frac{s-b}{4a}$ according to the canon; and $\frac{s-b}{4a}$ was found before to be 1273; divide this by b or 1142, and you will have $\frac{s-b}{4ab}$ or

or the time of the sound's ascent equal to 1.11 of a second: add this to the former time, to wit, 8.89 seconds, and the whole time will amount to 10 seconds, as it ought.

If any one be desirous to see, how the space through which a heavy body descends from rest in one second of time is deduced from *Huygens's* theorem above mentioned, it is done as follows: let l be the length of a pendulum that swings seconds; and then according to that theorem, the time wherein a heavy body descends from rest through a space equal to $\frac{1}{2}l$, will be to the time of one oscillation of the pendulum whose length is l , as the diameter of a circle is to the circumference; but the time of one oscillation of the pendulum l is one second by the supposition; or if we put s for the unknown space through which a heavy body falls from rest in one second of time, then the time of one oscillation of the pendulum l will be the same with the time of descent through the space s , and the proportion will now stand thus; the time of descent through $\frac{1}{2}l$ will be to the time of descent through s , as the diameter of a circle is to the circumference; and therefore the square of the former time will be to the square of the latter, as the square of the diameter is to the square of the circumference; but the square of the time of descent through $\frac{1}{2}l$ is to the square of the time of descent through s , as $\frac{1}{2}l$ is to s , according to what has been delivered above; therefore $\frac{1}{2}l$ is to s , as the square of the diameter of a circle is to the square of the circumference; but the diameter of a circle is to the circumference as 113 to 355, (see schol. 1. to art. 179;) and the square of the former is to the square of the latter as 12769 is to 126025; moreover l the length of a pendulum that swings seconds is found by experience to be 3 feet $8\frac{1}{2}$ lines, *Paris* measure, as above, that is, (by the introduction art. 23) 3.059028 feet; whence $\frac{1}{2}l = 1.529514$ of a foot; say then, as 12769 is to 126025, so is 1.529514 to 15.0957; and you will have $s = 15.0957$ French feet, or 16.1222 English feet.

Hence may be deduced by way of corollary, that as the square of the diameter of a circle is to the square of the circumference, so is half the length of any pendulum to the space through which a heavy body will fall from rest during the time of one oscillation of that pendulum.

T H E
SOLUTION OF TWO PROBLEMS

By Mr. ABRAHAM DE MOIVRE.

The curious and elegant solutions of the two following problems about proportionals were communicated by that most excellent Mathematician Mr. *de Moivre*, and his consent has been obtained for the printing of them. They are here inserted altogether such as Mr. *de Moivre* sent them, exactly in his own words.

P R O B L E M I.

The sum of four continual proportionals being given, as also the sum of their squares, to find the proportionals.

S O L U T I O N.

Let the four proportionals be x^3, xxy, xyy, y^3 : let their sum be $=a$, and the sum of their squares $=b$: we have therefore the two following equations;

$$x^3 + xxy + xyy + y^3 = a, \text{ and}$$

$$x^6 + x^4yy + xxy^4 + y^6 = b.$$

The first equation is reduced to this;

$$\overline{x + y \times xx + yy} = a: \text{ the second to}$$

$$\overline{xx + yy \times x^4 + y^4} = b.$$

The first equation reduced, determines me to suppose

$$x + y = z, \text{ and}$$

$$xx + yy = v:$$

Therefore the first equation is changed into

$$zv = a.$$

Now for the second, I say, since $x + y = z$, then squaring all,

$$xx + 2xy + yy = zz;$$

but $xx + yy$ has been supposed $=v$;

therefore $2xy + v = zz,$

and $2xy = zz - v,$

and $4xxyy = zz - 2vzz + vv:$

Again, since $xx + yy = v$, then squaring all,

$$x^4 + 2xxyy + y^4 = vv:$$

but the second equation reduced is

$$\overline{xx + yy \times x^4 + y^4} = b, \text{ or}$$

$$v \times x^4 + y^4 = b;$$

therefore

$$x^4 + y^4 = \frac{b}{v};$$

there-

therefore instead of the equation we had a little before, which was $x^4 + 2xxyy + y^4 = vv$, we may write

$$\frac{b}{v} + 2xxyy = vv, \text{ or}$$

$$2xxyy = vv - \frac{b}{v}, \text{ or}$$

$$4xxyy = 2vv - \frac{2b}{v}: \text{ but we had found before, that}$$

$$4xxyy = z^4 - 2vzz + vv; \text{ therefore we have now the}$$

equation $2vv - \frac{2b}{v} = z^4 - 2vzz + vv, \text{ or}$

$$vv - \frac{2b}{v} = z^4 - 2vzz;$$

But by the transformation of the first equation we have had $xv = a$;

therefore $v = \frac{a}{x};$

and this value of v being substituted, we shall have

$$\frac{aa}{zz} - \frac{2bz}{a} = z^4 - 2az, \text{ or}$$

$$z^6 + \frac{2b}{a} z^3 - 2az^3 = aa, \text{ which is properly a quadratic.}$$

Now z being found in that equation, then v will be found: when z and v are known, then x and y will be known; for 'tis plain that

$$x = \frac{1}{2}z + \frac{1}{2}\sqrt{2v - zz}, \text{ and}$$

$$y = \frac{1}{2}z - \frac{1}{2}\sqrt{2v - zz}; \text{ as will appear, if we consider}$$

that by supposition $x + y = z$, and

$$xx + yy = v;$$

Square the first of these two equations, and we shall have

$$xx + 2xy + yy = zz; \text{ but the second equation is}$$

$$xx + yy = v; \text{ from which taking their difference,}$$

$$2xy = zz - v; \text{ we have}$$

$$xx - 2xy + yy = v - zz + v = 2v - zz;$$

hence

$$x - y = \sqrt{2v - zz};$$

but

$$x + y = z;$$

therefore

$$2x = z + \sqrt{2v - zz},$$

and

and
$$x = \frac{1}{2}z + \frac{1}{2}\sqrt{2v - zz}.$$

Again, it appears that
$$2y = z - \sqrt{2v - zz};$$

therefore
$$y = \frac{1}{2}z - \frac{1}{2}\sqrt{2v - zz}.$$

Therefore the four proportionals are known.

PROBLEM 2.

The sum of five proportionals, and the sum of their squares being given, to find the proportionals.

SOLUTION.

Let the sum of the proportionals be a , and the sum of their squares b ; then we have these two equations;

$$x^4 + x^3y + xxyy + xy^3 + y^4 = a, \text{ and}$$

$$x^8 + x^6yy + x^4y^4 + xxy^6 + y^8 = b.$$

The first may be reduced to
$$\frac{x^5 - y^5}{x - y} = a;$$

the second to
$$\frac{x^{10} - y^{10}}{xx - yy} = b.$$

Divide the second of these equations by the first; then we shall have

$$\frac{x^5 + y^5}{x + y} = \frac{b}{a}; \text{ which equation being expanded, gives}$$

$$x^4 - x^3y + xxyy - xy^3 + y^4 = \frac{b}{a}.$$

Write under this, the first original equation, which was

$$x^4 + x^3y + xxyy + xy^3 + y^4 = a.$$

Let these two equations be added together; then we shall have

$$2x^4 + 2xxyy + 2y^4 = a + \frac{b}{a}, \text{ or}$$

$$x^4 + xxyy + y^4 = \frac{1}{2}a + \frac{\frac{1}{2}b}{a} = s,$$

(supposing for abbreviation, that $\frac{1}{2}a + \frac{\frac{1}{2}b}{a} = s$.)

But if, instead of adding those equations, we now subtract the uppermost from the lowermost, we shall have

$$2x^3y + 2xy^3 = a - \frac{b}{a}, \text{ or}$$

$$x^3y + xy^3 = \frac{1}{2}a - \frac{\frac{1}{2}b}{a}, \text{ or}$$

L 1

xyx

$xy \times \overline{xx + yy} = \frac{1}{2}a - \frac{\frac{1}{2}b}{a}$; which known quantities to shorten, we may call d : we have therefore the equation

$$xy \times \overline{xx + yy} = d.$$

Here I am naturally determined to suppose $xy = z$, and $xx + yy = v$; therefore instead of the last equation,

I have $zv = d$.

My business now is to express the other equation, $x^4 + xxxy + y^4 = s$ by means of z and v : but since $xx + yy = v$, then squaring all, we have

$$x^4 + 2xxxy + y^4 = vv; \text{ from which,}$$

subtracting $x^4 + xxxy + y^4 = s$, which we had before; then there remains

$$xxxy = vv - s:$$

But $xy = z$; therefore $xxxy = xz$, and therefore

$$xz = vv - s. \text{ But we had before, } zv = d:$$

Let z be exterminated; then we shall have

$$\frac{dd}{vv} = vv - s, \text{ or } dd = v^2 - svv.$$

Now v may easily be found in this equation; v being known, z will be known; both z and v being known, x and y will be known; for

$$x = \frac{1}{2} \sqrt{v + 2z} + \frac{1}{2} \sqrt{v - 2z}$$

$$y = \frac{1}{2} \sqrt{v + 2z} - \frac{1}{2} \sqrt{v - 2z}, \text{ as will appear if we con-}$$

sider that by supposition $xy = z$, and

$$xx + yy = v:$$

For doubling the first of these two equations, we shall have $2xy = 2z$; and this being added to the second, we have $xx + 2xy + yy = v + 2z$, therefore $x + y = \sqrt{v + 2z}$.

Again, if we subtract the equation $2xy = 2z$ from the other $xx + yy = v$, we shall have $xx - 2xy + yy = v - 2z$; wherefore $x - y = \sqrt{v - 2z}$. Now from the two equations,

$$x + y = \sqrt{v + 2z},$$

and $x - y = \sqrt{v - 2z}$, it is plain that

$$x = \frac{1}{2} \sqrt{v + 2z} + \frac{1}{2} \sqrt{v - 2z}$$

and $y = \frac{1}{2} \sqrt{v + 2z} - \frac{1}{2} \sqrt{v - 2z}.$

N. B.

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N. B. It perhaps may be proper for the sake of the young Analyst, to subjoin an example to each of the foregoing solutions communicated by Mr. De Moivre.

Example to the solution of the first problem.

Since $z^6 = \frac{2aa-2b}{a}z^3 + aa$, if for $\frac{aa-b}{a}$ we put d , then will $z^6 = 2dz^3 + aa$, and $z^3 = d + \sqrt{dd+aa}$: let $a=30$, $b=340$; then will $aa=900$, $aa-b=560$, $\frac{aa-b}{a}$ or $d=\frac{56}{3}$, $aa+dd=900+\frac{3136}{9}=\frac{11236}{9}$, $\sqrt{aa+dd}=\frac{106}{3}$, $d+\sqrt{aa+dd}$ or $z^3=54=27 \times 2=27c^3$ (putting c for the cube root of 2,) $z=3c$, $\frac{a}{z}$ or $v=\frac{30}{3c}=\frac{10}{c}=\frac{10cc}{c^3}=5cc$, $2v-zz=cc$, $\sqrt{2v-zz}=c$, $\frac{z+\sqrt{2v-zz}}{2}$ or $x=2c$, $\frac{z-\sqrt{2v-zz}}{2}$ or $y=c$;
whence the proportionals x^3 , xyy , xyy , y^3 ,
are $8c^3$, $4c^3$, $2c^3$, $1c^3$,
that is, 16 , 8 , 4 , 2 .

Example to the solution of the second problem.

Let $a=62$, $b=1364$; then $aa=3844$, $\frac{aa+b}{2a}$ or $s=42$, $\frac{aa-b}{2a}$ or $d=20$, $v^4=42vv+400$, $vv=21+\sqrt{841}=50=25 \times 2=25cc$ (putting c for the square root of 2,) $v=5c$, $\frac{d}{v}$ or $z=\frac{20}{5c}=\frac{4}{c}=\frac{4c}{cc}=2c$, $\sqrt{v+2z}=\sqrt{9c}=3\sqrt{c}$, $\sqrt{v-2z}=\sqrt{c}$, $\frac{1}{2}\sqrt{v+2z}+\frac{1}{2}\sqrt{v-2z}$ or $x=2\sqrt{c}$, $\frac{1}{2}\sqrt{v+2z}-\frac{1}{2}\sqrt{v-2z}$ or $y=\sqrt{c}$,
 $xx=4c$, $xy=2c$, $yy=c$;
and the proportionals x^4 , x^3y , xyy , xy^3 , y^4 ,
will be $16cc$, $8cc$, $4cc$, $2cc$, $1cc$,
or 32 , 16 , 8 , 4 , 2 .

T H E

ELEMENTS of ALGEBRA

B O O K V.

In what cases a problem may admit of many answers.

168. **I**T has already been observed, that if in any problem the number of independent conditions be equal to the number of unknown quantities, such a problem will admit but of one solution; or if it admits of more, they will however be so determined as to leave no room for arbitrary positions: but if the conditions be fewer in number than are the unknown quantities, those that are wanting may then be supplied by the Analyst himself at pleasure; and as there is infinite choice, it is no wonder if, in such a case a problem admits of an infinite number of answers, especially where fractions are taken into that number; but if the problem relates to whole numbers only, then the number of answers will sometimes be finite and sometimes infinite, as the nature of the problem will bear. This will be sufficiently illustrated by the two following examples.

E X A M P L E I.

Let it be required to find two numbers whose sum is equal to ten times their difference.

Here putting x and y for the two numbers sought, it is plain that in this case we have but one condition, and consequently but one equation, to wit, $x + y = 10x - 10y$, which equation being reduced, gives $x = \frac{11y}{9}$; and this is all the problem requires. Here then it is plain that the Analyst is entirely at liberty to substitute whatever whole number, mixt number, or proper fraction he pleases for y , provided he does but make $x = \frac{11y}{9}$; and the two quantities x and y will solve the problem.

As

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As for instance, let $\frac{1}{2}$ be put for y ; then will x or $\frac{11y}{9}$ be $\frac{11}{18}$, and those two fractions $\frac{11}{18}$ and $\frac{1}{2}$ or $\frac{9}{18}$ will solve the problem; for their difference is $\frac{1}{9}$, and their sum $\frac{10}{9}$. But if it be intended that x and y shall both be whole numbers, then such a whole number must be substituted for y as will admit of 9 for a divisor without a remainder: but of such whole numbers there is infinite choice, as 9, 18, 27, 36 &c; therefore this question is capable of an infinite number of answers, both in whole numbers and fractions.

EXAMPLE 2.

Let it now be required to find two numbers x and y , the product of whose multiplication is equal to ten times their difference.

Here the equation will be $y \cdot x = 10x - 10y$, which being reduced, gives $x = \frac{10y}{10-y}$. Here it is plain that y must be less than 10; for if y was equal to 10, the fraction $\frac{10y}{10-y}$ would be infinite, as will be shewn in another place; and if y be greater than 10, then $10-y$, and consequently $\frac{10y}{10-y}$ will be a negative quantity; whereas the problem may be supposed to relate to affirmative quantities only: however, as there is infinite choice of fractions between 0 and 10, and as any of these may be substituted for y , the problem will still be capable of an infinite number of solutions, if fractions may be admitted; but if it be required that x and y be both whole numbers, then there cannot be above nine such numbers that can be put for y ; nor perhaps all these neither, as remains in the next place to be shewn. Now to find what whole number being put for y will bring out x a whole number also, I reduce the quantity $\frac{10y}{10-y}$ to a more simple one, by dividing $10y$ by $10-y$, or rather by $-y+10$, beginning with $-y$ thus: $10y$ divided by $-y$ quotes -10 , which I put down in the quotient; then multiplying the divisor $-y+10$ by -10 the quotient, I find the product to be $+10y - 100$, which being subtracted from $10y$ the dividend, leaves 100 for a remainder; but not intending to carry on the division any further, I represent the rest of the quotient by the fraction $\frac{100}{10-y}$; so $x = \frac{100}{10-y} - 10$; therefore that x may be a whole number, it is necessary that $\frac{100}{10-y}$ be a whole number; but this will be impossible, unless $10-y$ be

be some one of the divisors of 100, I mean such a number as will divide 100 without remainder: I enquire therefore in the next place, how many such divisors 100 will admit of that are under 10; for so long as y is any thing, $10 - y$ must be less than 10; and I find four such divisors, to wit, 1, 2, 4 and 5; therefore if $10 - y$ be put equal to any of these, x or $\frac{100}{10 - y} - 10$ must come out a whole number; and it must also come out affirmative; for so long as $10 - y$ is greater than nothing and less than 10, $\frac{100}{10 - y}$ will always be greater than $\frac{100}{10}$, that is, than 10, and consequently $\frac{100}{10 - y} - 10$ or x will be affirmative. Let us

then suppose first, $10 - y = 1$, and we shall have $y = 9$, and $\frac{100}{10 - y}$ or $x = 90$. 2dly, if $10 - y = 2$, we shall have $y = 8$, and $x = 40$. 3dly, if $10 - y = 4$, we shall have $y = 6$, and $x = 15$. Lastly, if $10 - y = 5$, we shall have $y = 5$, and $x = 10$: therefore this question admits of 4 solutions in whole numbers, to wit, 90 and 9, 40 and 8, 15 and 6, and 10 and 5; all which equally answer the condition of the problem, as will appear upon tryal.

Thus having given the reader a taste of what he is to expect under this head, I shall now make some preparation for the solution of such problems as usually fall under this denomination.

DEFINITIONS.

169. If any one quantity can be measured or divided by another of the same kind without remainder, the former is called a multiple of the latter; as 12 is a multiple of 4, and 6 is a multiple of 3; for in this sense, every quantity is a multiple of itself, because every quantity measures itself: but if any one quantity can be so measured or divided by two or more others, it is then said to be a common multiple of those others; as 60 is a common multiple of the numbers 2, 3, 4, 5, 6.

COROLLARY.

Hence, if any two unequal quantities a and b be both multiples of a third, their sum $a + b$, and their difference $a - b$ or $b - a$, will also be multiples of the same; and if a and b be common multiples of any number of quantities, their sum and difference will also be common multiples of the same; for whatever quantities will measure both a and b , the same will measure both their sum and difference, (see art. 20, ax. 2d;) therefore

Art. 169, 170. *Lemmas relating to multiples and common measures.* 271
 fore *e converso*, whatever quantity or quantities *a* and *b* are multiples or common multiples of, the same will their sum and difference be multiples or common multiples of.

N. B. The following lemmas contain some curious properties, first of whole numbers, and then of fractions, worthy to be treated of as well for their own sakes, as for the use they have in the resolution of some of the following problems.

LEMMA I.

A THEOREM.

170. *Let a and b be any two quantities whatever whose least common multiple is c: I say then, that this least common multiple c shall measure any other common multiple d of the same quantities.*

If this be denied, it must then be allowed that after *c* has measured *d* as often as is possible, there must still remain a quantity as *e* less than *c*: be it so, and then *c* will measure $d - e$. Since then *a* and *b* both measure *c*, and *c* measures $d - e$, *a* and *b* will both measure $d - e$; therefore $d - e$ is a common multiple of *a* and *b*; but *d* is a common multiple of the same by the supposition; therefore both *d* and $d - e$ are common multiples of *a* and *b*; therefore their difference *e* is a common multiple of the same, by art. 169; but *e* is less than *c* by the supposition; therefore if *c* does not measure *d* without a remainder, it will be possible for two quantities *a* and *b* to have a common multiple less than their least; but this is impossible; therefore *c* does measure *d* without a remainder.
 Q. E. D.

COROLLARY.

If the demonstration here given had been applied to three or more quantities, it would equally have proved that their least common multiple will measure any other common multiple whatever of the same quantities.

SCHOLIUM.

This way of demonstrating the truth of a proposition by running it's contrary into an absurdity, or into an impossibility, or into a contradiction to something already established and demonstrated, was with a great deal of subtilty practiced by the Ancients whenever a more direct proof could not so easily be come at; and it is for that reason, that this sort of demonstrations is sometimes less embarrassed than those of the direct kind, where less subtilty is required: and certainly they are as firm, and as conclusive as any others whatever, as depending upon the same axioms and principles; and if they be not altogether so convincing, it must be
 because

because the machinery and contrivance of them does not so immediately appear, especially to young Mathematicians, who cannot as yet be supposed to be acquainted with all the different modes of thinking and reasoning upon quantity: therefore whenever I can, from a doubtful proposition, by just reasoning, deduce another that is apparently false, it may serve for an infallible mark to shew, that the first position from whence the deduction was made, was not true; for truth can lead to nothing but truth.

LEMMA 2.

A PROBLEM.

171. *It is required, having given any two unequal numbers a and b , to find their least common multiple.*

SOLUTION.

As ab , the product of a and b multiplied together, is a common multiple of both, it may be divided by their least common multiple without a remainder, by art. 170: imagine it then to be so divided, and let the quotient be c ; then *e converso*, if the product ab be divided by c , the quotient $\frac{ab}{c}$ will be the least common multiple of a and b : whence

I infer, that $\frac{a}{c}$ and $\frac{b}{c}$ will be whole numbers; for else, how should $\frac{ab}{c}$ be any common multiple of a and b , much less their least common multiple? but if $\frac{a}{c}$ and $\frac{b}{c}$ be whole numbers, then c must be a common measure of a and b ; and I say in the next place, that it must be their greatest common measure; for should any number greater than c , suppose d , measure them both; since $\frac{a}{d}$ and $\frac{b}{d}$ would be whole numbers, $\frac{ab}{d}$ would be a common multiple of a and b less than $\frac{ab}{c}$, because d is greater than c , and so a and b would have a common multiple less than their least, which is absurd: therefore, *If a and b , the two numbers first proposed, be multiplied together, and their product divided by their greatest common measure, the quotient will be their least common multiple. Q. E. I.*

Thus if the numbers 12 and 15 be proposed, whose greatest common measure is 3; the product of their multiplication is 180, and this divided by 3 gives 60 for their least common multiple.

COROL-

COROLLARY 1.

Hence if the numbers given be prime to each other, that is, are such as will admit of no common measure greater than unity, the product of their multiplication will be their least common multiple: for in this case,

$$\frac{ab}{c} = \frac{ab}{1} = ab.$$

COROLLARY 2.

If the fraction $\frac{a}{b}$ be reduced to it's least terms $\frac{d}{e}$; I say then that ae or bd (for they are equal) will be the least common multiple of a and b : for in this case, if c be the greatest common measure of a and b , we shall have $d = \frac{a}{c}$, and $e = \frac{b}{c}$, and therefore ae or bd will be $\frac{ab}{c}$.

LEMMA 3.

A PROBLEM.

172. It is required, having given three or more numbers, to find their least common multiple.

Let the numbers given be a, b, c and d , and the solution will be as follows: First by the foregoing article find the least common multiple of a and b , and call it e ; then find the least common multiple of e and c , and call it f ; then find the least common multiple of f and d , and call it g : I say then that the number g last found will be the least common multiple of the numbers a, b, c and d . For first, that g will be a common multiple of those numbers I thus demonstrate: a and b must measure e , their least common multiple; and e will measure f , the least common multiple of e and c ; and f will measure g , the least common multiple of f and d ; therefore both a and b will measure g : again, c will measure f , the least common multiple of e and c ; and f measures g as before; therefore c measures g ; but d measures g *ex hypothesi*; therefore all the given numbers a, b, c and d measure g ; therefore g is a common multiple of them all. I say in the next place that g is their least common multiple: for if it be possible, let h be a less, that is, let all the numbers a, b, c and d measure a number h that shall be less than g ; and since h is a common multiple of a and b *ex hypothesi*, and e is their least common multiple, e must measure h by art. 170; but c measures h *ex hypothesi*; therefore since both e and c measure h , their least common multiple f must also measure h ; but d measures h *ex hypothesi*; therefore since both f and d measure h , their least common multiple g must also measure h , that is, a

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greater quantity must measure a less, which is absurd; therefore g is the least common multiple of the numbers given. *Q. E. I.*

Of this solution take the following example. Let it be required to find the least common multiple of the numbers 8, 10, 12 and 14: now according to the last article, the least common multiple of 8 and 10 is 40; and the least common multiple of 40 and 12 is 120; and the least common multiple of 120 and 14 is 840; therefore 840 is the least common multiple of the numbers 8, 10, 12 and 14.

LEMMA 4.

A PROBLEM.

173. *Let a and b be any two unequal quantities whereof c and d are multiples respectively: It is required to find as many more multiples as we please of the same quantities with the same difference, that is, that a 's multiple may always exceed that of b by $c-d$, if c be greater than d ; or else that b 's multiple may always exceed that of a by $d-c$, if d be greater than c .*

SOLUTION.

Let $c+e$ and $d+e$ represent any two multiples of this kind, that is, whose difference is $c-d$ or $d-c$; then since both c and $c+e$ are multiples of a , e will also be a multiple of a , by art. 169; for the same reason, since both d and $d+e$ are multiples of b , e will be a multiple of b ; therefore *If e be any common multiple of a and b , $c+e$ and $d+e$ will be multiples of the same, and such as were required, and not otherwise.* Let now e be the least common multiple of a and b ; then it is plain that $c+e$ and $d+e$ will be the next in their kind greater than c and d , and that $c+2e$ and $d+2e$ will be the next greater than $c+e$ and $d+e$, and so on ad infinitum: on the other hand $c-e$ and $d-e$ will be the next in their kind less than c and d , and $c-2e$ and $d-2e$ the next less than $c-e$ and $d-e$, and so on, so long as the subtraction can be continued.

Whence it follows, that *If e be the least common multiple of a and b , and if either c or d or both be less than e , those multiples c and d will be the least in their kind: as for example, let $a=4$ and $b=6$, then will 8 and 18 be two multiples of a and b , whose difference is 10; and they will be the least in their kind, because the least common multiple of a and b is 12, and 8 is less than 12: but if other multiples with the same difference be required, they will easily be had by a continual addition of the least common multiple 12; as 20 and 30, 32 and 42, 44 and 54, &c ad infinitum.*

LEMMA 5.

LEMMA 5.

A THEOREM.

174. *Let a and b be any two quantities whose greatest common measure is c: I say then that if any two unequal multiples of a and b be taken and compared, their difference can never be less than the greatest common measure c.*

For since, by the supposition, c measures both a and b, it must not only measure all their multiples, but also the differences of those multiples; and therefore no difference can be less than c. Q. E. D.

LEMMA 6.

A PROBLEM.

175. *It is required, having given two numbers a and b, whereof a is supposed to be the greater, to find two unequal multiples of these numbers whose difference shall be the least possible, that is by the last article, whose difference shall be the greatest common measure of a and b.*

SOLUTION.

Let the greater number be 270, and the less 112, whose greatest common measure found by art. 21, is 2: it is required then to find a multiple of 270, and another of 112, whose difference shall be 2, and that, whichever of the two multiples is intended for the greater. Here $a = 270$, and $b = 112$, which two equations I put down one under the other in the form following:

$$\begin{array}{l} \text{1st, } 1a - 0b = +270, \\ \text{2d, } 0a - 1b = -112, \end{array}$$

where $0a$ and $-0b$ are only put down to make these equations of the same form with others, which I derive from them thus: first I divide 270 the absolute term of the first equation by 112 the absolute term of the second, without having regard to it's sign, and the nearest quotient too little is 2; therefore I multiply the second equation by 2, and the product is $0a - 2b = -224$; to this equation I add the first, $1a - 0b = +270$, and the sum is $a - 2b = +46$, which I put down under the other two for a third equation, as you will find in the table annexed: this done, I again divide 112 the absolute term of the second equation, without regarding it's sign, by 46 the absolute term of the third, and the nearest quotient too little is 2; therefore I multiply the third equation by 2, and the product is $2a - 4b = +92$; to this I add

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the equation next above it, *viz.* $0a - 1b = -112$, and the sum makes a fourth equation, to wit, $2a - 5b = -20$, which I put down under the rest: then I divide 46 the absolute term of the third equation by 20 the absolute term of the fourth, and still the nearest quotient too little is 2; therefore to twice the fourth equation I add the third, and so have a fifth, to wit, $5a - 12b = +6$: this being put down, I divide 20 the absolute term of the fourth equation by 6 the absolute term of the fifth, and the nearest quotient too little is 3; therefore to thrice the fifth equation I add the fourth, and so have a sixth, to wit, $17a - 41b = -2$: lastly I divide 6 the absolute term of the fifth equation by 2 the absolute term of the sixth, and the true quotient is 3, but the nearest too little is 2: now if to three times the sixth equation I should add the fifth, I should have $56a - 135b = 0$; but because I do not want an equation whose absolute term is nothing, but only an equation whose absolute term is the least possible, instead of the true quotient 3 I use 2 the nearest quotient too little, as I have all along hitherto done, and so adding the fifth equation to twice the sixth, I have a seventh, to wit, $39a - 94b = +2$.

Having thus finished my series of equations, it is evident that the two last equations will solve the problem; for if the multiple of a is to be greater than that of b , $39a$ and $94b$ will be the multiples required, because $39a - 94b = +2$; but if the multiple of b is intended to be greater than that of a , then $17a$ and $41b$ will be the multiples sought, because $17a - 41b = -2$, and consequently $41b - 17a = +2$. This will further be confirmed upon tryal; for if a be 270, and b 112, we shall have $39a = 10530$, $94b = 10528$, $17a = 4590$, $41b = 4592$.

Equ. 1, $1a - 0b = +270$.

2, $0a - 1b = -112$.

3, $a - 2b = +46$.

4, $2a - 5b = -20$.

5, $5a - 12b = +6$.

6, $17a - 41b = -2$.

7, $39a - 94b = +2$.

Lastly, that this series of absolute terms in the equations above described will always terminate at last in the greatest common measure of a and b , will be evident from the very method of finding the greatest common measure, laid down in art. 21, which is the very same with that whereby the absolute terms are found: for to find the greatest common measure of the numbers 270 and 112, 270 must be divided by 112, 112 by 46, 46 by 20, 20 by 6, and 6 by 2, and the series 270, 112, 46, 20, 6, 2 terminates in the greatest common measure: but this series is altogether the same with the series of absolute terms found above, except that

that there, the greatest common measure is twice repeated, which arises from hence, that the true quotient at last was not taken, but only the nearest quotient too little; therefore one of the two last equations will always solve the problem. *Q. E. D.*

COROLLARY 1.

Since the fraction $\frac{a}{b}$ or $\frac{270}{112}$, when reduced to it's least terms, is $\frac{135}{56}$, it follows by art. 171, that $56a = 135b$ is the least common multiple of a and b , and consequently that $56a - 135b = 0$ is the simplest equation that can have 0 for it's absolute term. Let now this equation be continually added to either of the two last equations which solves the problem, and you will have an infinite number of solutions more of the same problem, that is, you will have an infinite number of equations whose absolute terms are $+$ or -2 : thus if to the last equation $39a - 94b = 2$ be added the equation $56a - 135b = 0$, you will have $95a - 229b = 2$; and if again be added the same equation, you will have $151a - 364b = 2$, and so on: again, if to the last equation but one, to wit, $17a - 41b = -2$ be added the equation $56a - 135b = 0$, you will have $73a - 176b = -2$; and if again be added the same equation, you will have $129a - 311b = -2$, and so on *ad infinitum*.

COROLLARY 2.

The absolute terms of the equations above described will always be affirmative and negative alternately, and the absolute term of the last equation will be negative or affirmative according as the number of equations is even or odd: thus in the foregoing table the absolute terms were $+270$, -112 , $+46$, -20 , $+6$, -2 , $+2$, and the last term was $+2$ as possessing an odd place.

SCHOLIUM.

If the foregoing problem be stated algebraically, it will stand thus; *It is required, having given two numbers a and b, whose greatest common measure is c, to find other two numbers x and y of such a nature, that being multiplied by the numbers a and b respectively, the difference of the products ax - by may be found equal to + or - c.*

Here then we have but one equation for finding two unknown quantities, and that is the reason why the problem admits of an infinite number of solutions, to wit, by a continual addition of the equation $56a - 135b = 0$, as in corollary 1: but though this equation may be continually added to any of the equations exhibited in the foregoing table, and that *ad infinitum*, yet it cannot be subtracted from any of them; which

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is a manifest argument that those equations are the most simple of their kind, that is, the most simple of all others that have the same absolute terms; see art. 173: this will be more universally demonstrated in art. 177. In the mean time, in order to form general conclusions concerning the nature of those equations, it will be proper in all the equations except the two first, to denote the coefficients of a and b at least, in general terms, thus: in the third equation we have $1a - 2b = +46$; make $p = 1$ and $P = 2$, and you will have $pa - Pb = +46$: in like manner in the fourth equation, if q be made equal to 2 and Q equal to 5, you will have $qa - Qb = -20$; and so on. See the equations according to this way of notation:

$$\begin{array}{ll} \text{Equ. 1,} & 1a - 0b = +270. \\ & 2, \quad 0a - 1b = -112. \\ & 3, \quad pa - Pb = +46. \\ & 4, \quad qa - Qb = -20. \\ & 5, \quad ra - Rb = +6. \\ & 6, \quad sa - Sb = -2. \\ & 7, \quad ta - Tb = +2. \end{array}$$

LEMMA 7.

A THEOREM.

176. *If in any of the equations exhibited in the last article, the coefficients of a and b be multiplied crosswise into the coefficients of a and b in the equation next it, whether above or below, b 's coefficient being considered as affirmative; I say then that the difference of the two products thence arising will always be equal to unity.*

As for example, let the two last equations be taken, and let t the coefficient of a in the last equation be multiplied into S the coefficient of b in the last but one; let also T the coefficient of b in the last equation be multiplied into s the coefficient of a in the last but one: I say then that the difference betwixt the two products tS and Ts will be equal to unity; and that the same will happen throughout the whole series, provided that the equations made choice of be such as lie nearest one another.

To demonstrate this, let the absolute term of the last equation but two be divided by the absolute term of the last but one, and let the nearest quotient too little be n ; then it is plain from the nature of these equations, that $nsa + ra$ will be equal to ta , and that $-nSb - Rb$ will be equal to $-Tb$; therefore $ns + r = t$, and $nS + R = T$. Put now instead of t and T their values thus found, and you will have $tS = nSs + Sr$, and $Ts = nSs + sR$; therefore the difference between tS and Ts will be the same with the difference between sR and Sr ; and for the

Art. 176, 177. WHOLE NUMBERS AND FRACTIONS. 279

the same reason, the difference between sR and Sr is the same with the difference between rQ and Rq ; and this again the same with the difference between qP and Qp , &c: therefore if this difference in any two nearest equations be known, it will every where be known, as being every where the same; but 1 the coefficient of a in the first equation being multiplied into 1 the affirmative coefficient of b in the second, gives 1 for the product; and moreover 0 the coefficient of b in the first equation being multiplied into 0 the coefficient of a in the second gives 0 for the product, and the difference betwixt the products 1 and 0 is 1; therefore the difference of the products every where will be 1. Q. E. D.

LEMMA 8.

A THEOREM.

177. *Supposing all things as in the two last articles, if the coefficients of a and b in the several equations exhibited in the scholium to the hundred seventyfifth article be turned into fractions, by making the coefficients of b the numerators, and those of a the denominators; I say then that the fractions $\frac{0}{1}, \frac{1}{0}, \frac{P}{p}, \frac{Q}{q}, \frac{R}{r}, \frac{S}{s}, \frac{T}{t}$ will all be in their least terms.*

For if the fraction $\frac{T}{t}$ for instance, be not in it's least terms, the numbers T and t must admit of some common measure greater than unity: be it so, and let this common measure be called m ; then since the number m measures both the numbers T and t , it must also measure both the products Ts and tS , and consequently their difference; but the difference of these two products is unity, by the last article; therefore m a number greater than unity must measure unity, which is absurd; therefore the fraction $\frac{T}{t}$ is in it's least terms: and the same may be affirmed of all the rest. Q. E. D.

COROLLARY.

If the absolute term of the last equation but one, which is the greatest common measure of a and b , be divided by the absolute term of the last, which is also the same common measure, the quotient will be 1: therefore if the last equation be multiplied by 1, and to the product be added the last equation but one, or which is the same thing, if the two last equations be added together, there will arise an equation of this form, $va - Vb = 0$, the two absolute terms destroying each other;

other; whence $\frac{V}{v} = \frac{a}{b}$; but the fraction $\frac{V}{v}$ is in it's least terms by this proposition, as being produced after the same manner as the rest; therefore $av = bV$ is the least common multiple of a and b , and the equation $av - bV = 0$ is the simplest that can have 0 for it's absolute term: but av and bV are greater multiples of a and b than can be found in any of the equations exhibited in art. 175, as is evident from the *genesis* of those equations; therefore the equation $av - bV = 0$ cannot be subtracted from any of those equations, at least without destroying the coefficient of a ; therefore those equations are the simplest of their kind.

L E M M A 9.

A T H E O R E M.

178. *Supposing all things as in the last article, I say that the fractions*
 $\frac{0}{1}, \frac{1}{0}, \frac{P}{p}, \frac{Q}{q}, \frac{R}{r}, \frac{S}{s}, \frac{T}{t}$ *are of such a nature as to be alternately*
less and greater than the fraction $\frac{a}{b}$, *but yet so as constantly to converge towards it.*

Thus the fraction $\frac{0}{1}$ is infinitely less than the fraction $\frac{a}{b}$, and the fraction $\frac{1}{0}$ is infinitely greater; for the less the denominator of any fraction is, *cæteris paribus*, the greater will be the fraction; therefore if the denominator be nothing, or infinitely small, the value of such a fraction must be infinitely great: thus again the fraction $\frac{P}{p}$ is less than $\frac{a}{b}$, and the fraction $\frac{Q}{q}$ greater, and so on alternately. I say further, that the fraction $\frac{Q}{q}$ approaches nearer to $\frac{a}{b}$ than doth the fraction $\frac{P}{p}$, and the fraction $\frac{R}{r}$ nearer than $\frac{Q}{q}$, and so on.

To demonstrate this, let the equation $ca - Cb = \pm d$ be a general representative of all the equations in art. 175; and dividing both sides by bc , we shall have $\frac{a}{b} - \frac{C}{c} = \pm \frac{d}{bc}$; whence I infer 1st, that the fraction $\frac{a}{b}$ will be greater or less than the fraction $\frac{C}{c}$, according as the absolute

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Art. 178, 179. OF CONVERGING FRACTIONS. 281

lute term d is affirmative or negative; 2^{dly}, that the less the absolute term d is, or the greater the denominator c is, the nearer will the fraction $\frac{C}{c}$ approach towards the fraction $\frac{a}{b}$: but throughout the whole series of equations in art. 175, the absolute terms are alternately affirmative and negative; therefore the series of fractions deduced from those equations will be of such a nature, that the fractions will be alternately less and greater than the fraction $\frac{a}{b}$: it is also evident from the nature of the equations above-mentioned, that the series of terms represented by d will be constantly diminishing, and that the terms represented by c will be constantly increasing; and therefore upon both these accounts, the fractions represented by $\frac{C}{c}$ must converge on both sides towards the fraction $\frac{a}{b}$.

COROLLARY.

Therefore if any two fractions that are nearest to each other be taken, as $\frac{T}{t}$ and $\frac{S}{s}$, the fraction $\frac{a}{b}$ must lie between them, but so as to lie nearer to the fraction $\frac{T}{t}$ than to the fraction $\frac{S}{s}$, otherwise the series would not converge.

LEMMA 10.

A THEOREM.

179. *Supposing all things as in the foregoing articles, I say that the fractions $\frac{P}{p}, \frac{Q}{q}, \frac{R}{r}, \frac{S}{s}, \frac{T}{t}$ exhibit the fraction $\frac{a}{b}$ more accurately than any other fractions whatever of less denominations can do; and particularly I say that the fraction $\frac{T}{t}$ approaches nearer to the fraction $\frac{a}{b}$ than any fraction can do whose denominator is less than t .*

If this be denied, let $\frac{C}{c}$ be a fraction whose denominator c is less than t ; first then I affirm that $\frac{C}{c}$ cannot be equal to $\frac{T}{t}$; for the fraction $\frac{T}{t}$ is in it's least terms already, by art. 177; therefore if the fraction $\frac{C}{c}$

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was equal to the fraction $\frac{T}{t}$, the denominator c must be equal to, or some multiple of the denominator t , whereas c is less than t by the supposition: now since the fraction $\frac{a}{b}$ lies between the two fractions $\frac{T}{t}$ and $\frac{S}{s}$, but nearer to $\frac{T}{t}$ by the last article, it follows that if the fraction $\frac{C}{c}$ approached nearer to $\frac{a}{b}$ than doth the fraction $\frac{T}{t}$, it must lie also between $\frac{T}{t}$ and $\frac{S}{s}$; but the difference between $\frac{T}{t}$ and $\frac{S}{s}$ is $\frac{1}{ts}$, by art. 176; and the difference between $\frac{C}{c}$ and $\frac{S}{s}$ is $\frac{Cs - cS}{cs}$, or $\frac{cS - Cs}{cs}$; and this difference cannot be less than $\frac{1}{cs}$: since then $\frac{1}{cs}$ is greater than $\frac{1}{ts}$ because c is less than t , it follows that the difference between $\frac{S}{s}$ and $\frac{C}{c}$ is greater than the difference between $\frac{S}{s}$ and $\frac{T}{t}$; therefore the fraction $\frac{C}{c}$ cannot lie between the two fractions $\frac{S}{s}$ and $\frac{T}{t}$, and consequently cannot approach so near the fraction $\frac{a}{b}$ as doth the fraction $\frac{T}{t}$. Q. E. D.

S C H O L I U M I.

In computing the equations in the 175th article, the quotients used for that purpose were 2, 2, 2, 3, 2, the last quotient 3 being diminished by unity to give such a solution of the problem there considered as was necessary to answer the end for which it was proposed; but had the true quotient 3 been taken, the last equation would have been $56a - 135b = 0$, as was there observed; and the fraction derived from it would have been $\frac{135}{56}$, which is nothing else but the fraction $\frac{a}{b}$ in its least terms; therefore if the last quotient be taken true, the fractions derived from those equations, to wit, $\frac{2}{1}$, $\frac{5}{2}$, $\frac{12}{5}$, $\frac{41}{17}$, $\frac{135}{56}$ will not only converge on both sides towards the intermediate fraction $\frac{a}{b}$, but at last will actually terminate in it.

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As to the practice of computing these fractions, there is no necessity of introducing the terms a and b into the series, and much less of computing the equations in order to deduce the fractions from them; for as the terms of these fractions are nothing else but the coefficients of a and b , they may be obtained much easier by themselves, in the manner following: Let the fraction $\frac{270}{112}$ be proposed, whose numerator and denominator are the terms a and b , as in art. 175, and let it be required to find other fractions of lesser denominations which, though not equal to the fraction proposed, shall nevertheless come as near it as any particular purpose requires, and nearer than any other fraction of a less denomination than these can: here the greater number 270 divided by the less 112 quotes 2, and there remains 46; again, 112 divided by 46 quotes 2, and there remains 20; 46 divided by 20 quotes 2, and there remains 6; 20 divided by 6 quotes 3, and there remains 2; and lastly 6 divided by 2 quotes 3, and there remains 0; therefore the quotients arising from this continual division are 2, 2, 2, 3, 3. These quotients being thus obtained, in the next place I assume two fractions $\frac{0}{1}$ and $\frac{1}{0}$ to begin with, and multiplying 1 and 0, the terms of the latter fraction, by the first quotient 2, the products are 2 and 0, to which I add 0 and 1, the terms of the first fraction, and so have 2 and 1 for the terms of my first finite fraction; therefore my first finite fraction is $\frac{2}{1}$; these terms 2 and 1 I multiply by my second quotient 2, and the products are 4 and 2, to which adding 1 and 0, the terms of the fraction immediately going before, I have $\frac{5}{2}$ for my next fraction; these terms 5 and 2 I multiply by my third quotient 2, and the products are 10 and 4, which with 2 and 1, the terms of the fraction immediately before it, give $\frac{12}{5}$ for my next fraction; these terms 12 and 5 I multiply by my fourth quotient 3, and the products are 36 and 15, which with 5 and 2, the terms of the next antecedent fraction, give $\frac{41}{17}$ for the terms of my next fraction; lastly I multiply 41 and 17 by my last quotient 3, and the products are 123 and 51, which with 12 and 5, the terms of the fraction before it, give $\frac{135}{56}$ for my last fraction; and so the fractions thus obtained are $\frac{2}{1}$, $\frac{5}{2}$, $\frac{12}{5}$, $\frac{41}{17}$, $\frac{135}{56}$. Where it may be observed, that if

I had placed the fraction $\frac{1}{0}$ before the fraction $\frac{0}{1}$, and then had derived the rest from these as before, the fractions would have come out inverted, to wit, $\frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{17}{41}, \frac{56}{135}$; and these fractions would have converged towards, and at last would have terminated in the intermediate one $\frac{b}{a}$, with this difference however, that now the first fraction $\frac{1}{2}$ would have been greater than $\frac{b}{a}$, the next $\frac{2}{5}$ less, and so on alternately.

The better to see the approximation of these fractions, I reduce them all to other fractions whose common numerator is 270, thus: as 2 the numerator of the first fraction is to 1 the denominator, so is 270 the new numerator of the same fraction to 135 the new denominator; whence $\frac{2}{1} = \frac{270}{135}$, which differs no more from $\frac{270}{112}$ than might be expected, considering the simplicity of the fraction $\frac{2}{1}$: again, as 5 the numerator of the next fraction is to 2 the denominator, so is 270 to 108; therefore $\frac{5}{2} = \frac{270}{108}$, which differs much less from $\frac{270}{112}$ than did the former fraction $\frac{2}{1}$: 3dly, as 12 the numerator of the next fraction is to 5 the denominator, so is 270 to $112\frac{1}{2}$; therefore $\frac{12}{5} = \frac{270}{112\frac{1}{2}}$, which is very near $\frac{270}{112}$: 4thly, as 41 the numerator of the next fraction is to 17 the denominator, so is 270 to $111\frac{39}{41}$ or $112 - \frac{1}{20}$ nearly; therefore $\frac{41}{17} = \frac{270}{112 - \frac{1}{20}}$: lastly, as 135 the numerator of the last fraction is to 56 the denominator, so is 270 to 112; therefore $\frac{135}{56} = \frac{270}{112}$.

For another example of this sort of approximation, take the proportion of the circumference of a circle to it's diameter, which according to *Van Ceulen's* numbers when abridged, is that of 314159265359 to 100000000000. Here making $a = 314159265359$, and $b = 100000000000$, let it be required to find a fraction in more simple terms that shall nearly express the fraction $\frac{a}{b}$, that is, more nearly than any other fraction of a less

Art. 179. TO A GIVEN DEGREE OF EXACTNESS. 285

less denomination; and the process will be as follows: 1st, a divided by b quotes 3, and there remains $c=14159265359$: 2dly, b divided by c quotes 7, and there remains $d=885142487$: 3dly, c divided by d quotes 15, and there remains $e=882128054$: 4thly, d divided by e quotes 1, and there remains $f=3014433$: 5thly, e divided by f quotes 292, and there remains $g=1913618$. This last quotient 292 being so very large, I conclude that the four first quotients 3, 7, 15 and 1 will give the last fraction accurately enough, since its terms must be multiplied by no less a number than 292 to obtain another that is more so. Using then the four first quotients 3, 7, 15, 1, I proceed as in the foregoing example,

and so find four fractions $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$ and $\frac{355}{113}$, which last fraction will be found exact enough for all purposes; for since the circumference

of a circle is to the diameter as a to b , and the fraction $\frac{a}{b}$ is nearly equal to $\frac{355}{113}$, it follows that the circumference is to the diameter as 355

to 113 nearly, or as $\frac{355}{113}$ is to 1, or (reducing the fraction to decimals) as 3.1415929 to 1, or lastly as 31415929 to 10000000, the antecedent of which proportion is true to 7, and almost to 8 places; for the proportion ought to be as 31415927 to 10000000.

Artists for the better remembering the proportion betwixt the diameter and circumference of a circle, as we have here established it, viz. 113 to 355, are used to conceive it thus: put down the three first odd numbers 1, 3 and 5, writing down every number twice, or at least imagine them to be put down thus, 113355; then will the three first numbers 113 belong to the diameter, and the three last, to wit, 355 to the circumference.

SCHOLIUM 2.

The fractions treated of in the last scholium may be called primary or principal fractions, all converging to, and at last meeting in the intermediate fraction $\frac{a}{b}$; but if it be required that the series be compleat, other fractions called secondary ones must be inserted between the primary ones, when the quotient is greater than unity, by changing the multiplication in the last scholium into a continual addition: in order to do this, it will be proper to make two rows of fractions, the upper to contain all the fractions that are greater than $\frac{a}{b}$, and the lower

to

to contain all those that are less, so as that $\frac{1}{0}$ may be the first fraction in the upper row, and $\frac{0}{1}$ the first in the lower; then making use of the first example in the foregoing scholium, where the quotients were 2, 2, 2, 3, 3, because the equations in the 175th article were particularly adapted to that case, add the terms of the fraction $\frac{1}{0}$ twice to the terms of the fraction $\frac{0}{1}$, because the first quotient was 2, and you will have two fractions arising from that continual addition, to wit, $\frac{1}{1}$ and $\frac{2}{1}$ to be placed in the lower row: add the terms of this last fraction $\frac{2}{1}$ twice to the terms of the fraction $\frac{1}{0}$, because the second quotient was 2, and you will have two fractions $\frac{3}{1}$ and $\frac{5}{2}$ to be placed in the upper row: add the terms of this last fraction $\frac{5}{2}$ twice to the terms of the last fraction $\frac{2}{1}$ in the lower row, because the third quotient was 2, and you will have the fractions $\frac{7}{3}$ and $\frac{12}{5}$ to be placed in the lower row: add the terms of this last fraction $\frac{12}{5}$ thrice to the terms of the last fraction $\frac{5}{2}$ in the upper row, because the fourth quotient was 3, and you will have $\frac{17}{7}$, $\frac{29}{12}$ and $\frac{41}{17}$ to be placed in the upper row: add the terms of this last fraction $\frac{41}{17}$ thrice to the last fraction $\frac{12}{5}$ in the lower row, because the last quotient was 3, and you will have $\frac{53}{22}$, $\frac{94}{39}$ and $\frac{135}{56}$ to be placed in the lower row: and thus you will have finished the whole series, or rather two serieses of fractions, whereof the upper series consists of fractions greater than $\frac{a}{b}$, and the lower of fractions that are less, as appears by the following table:

Fractions

Fractions greater than the true, $\frac{1}{0} \quad \frac{3}{1} \quad \frac{5}{2} \quad \frac{17}{7} \quad \frac{29}{12} \quad \frac{41}{17}$

Fractions less than the true, $\frac{0}{1} \quad \frac{1}{1} \quad \frac{2}{1} \quad \frac{7}{3} \quad \frac{12}{5} \quad \frac{53}{22} \quad \frac{94}{39} \quad \frac{135}{56}$

The last fraction in the lower row, to wit $\frac{135}{56}$, is equal to $\frac{a}{b}$, and therefore, properly speaking, belongs to neither row: every fraction in the upper series approaches nearer to $\frac{a}{b}$ than any other fraction in simpler terms can do that is greater than $\frac{a}{b}$, and every fraction in the lower series approaches nearer it than any other in simpler terms can do that is less than $\frac{a}{b}$; but it sometimes happens that a fraction on one side shall not approach so near it as a fraction on the other, though in more simple terms: thus the fraction $\frac{17}{7}$ in the upper series differs more from $\frac{270}{112}$ or $\frac{a}{b}$ on one side, than $\frac{12}{5}$ doth on the other; for the excess of $\frac{17}{7}$ above $\frac{270}{112}$ is $\frac{1}{56}$ or $\frac{10}{560}$, whereas the excess of $\frac{270}{112}$ above $\frac{12}{5}$ is $\frac{6}{560}$; but this never happens when the fraction that hath the greater denominator is a primary fraction.

All I have delivered in this scholium will best be comprehended by treating the equations in the 175th article after the same manner as the fractions here, that is, by turning the multiplication there made use of into a continual addition; but for the better distinguishing these equations, make two columns, placing all the equations whose absolute terms are negative in the left-hand column, and all those whose absolute terms are affirmative on the right, so as that the first equation on the left hand may be $0a - 1b = -112$, and the first on the right hand may be $1a - 0b = +270$: this done, you may proceed as follows; 1st, add the equation $0a - 1b = -112$ twice to the equation $1a - 0b = +270$, and you will have two equations to be placed in the right-hand column, to wit, $1a - 1b = +158$, and $1a - 2b = +46$: 2dly, add this last equation $1a - 2b = +46$ twice to the equation $0a - 1b = -112$, and you will have two equations to be placed in the left-hand column, to wit, $1a - 3b = -66$, and $2a - 5b = -20$: 3dly, add this last equation twice to the equation $1a - 2b = +46$, and you will have two equations to be placed on the right hand, to wit, $3a - 7b = +26$,

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$-7b = +26$, and $5a - 12b = +6$: 4thly, add this last equation thrice to the equation $2a - 5b = -20$, and you will have three equations to be placed on the left hand, to wit, $7a - 17b = -14$, $12a - 29b = -8$, and $17a - 41b = -2$: 5thly, add this last equation thrice to the equation $5a - 12b = +6$, and you will have three equations to be placed in the right-hand column, to wit, $22a - 53b = +4$, $39a - 94b = +2$, and $56a - 135b = 0$. Thus you will have finished your serieses of equations, from whence the two serieses of fractions already computed have been derived.

$$0a - 1b = -112.$$

$$1a - 3b = -66.$$

$$2a - 5b = -20.$$

$$7a - 17b = -14.$$

$$12a - 29b = -8.$$

$$17a - 41b = -2.$$

$$1a - 0b = +270.$$

$$1a - 1b = +158.$$

$$1a - 2b = +46.$$

$$3a - 7b = +26.$$

$$5a - 12b = +6.$$

$$22a - 53b = +4.$$

$$39a - 94b = +2.$$

$$56a - 135b = 0.$$

Much more might be produced upon this head; but I fear most of my readers think I have been too prolix already, especially those, who being less conversant in numbers, cannot so easily be made sensible of the usefulness of this doctrine.

This way of approximation to fractions of higher denominations is owing to the sagacity of the ingenious Mr. *Cotes* late Professor of Astronomy and experimental Philosophy in this University, whose great and happy genius to all mathematical enquiries made him conceive and treat the most difficult subjects with the utmost simplicity and ease; and whose posthumous pieces not long ago published by his worthy successor Dr. *Smith*, will ever be esteemed and admired, so long as a good taste and sound judgment shall have any being amongst mankind.

This subject had been handled before, though in a very different way, by the learned Dr. *Wallis*, and Monsieur *Huygens*; but I am of opinion, that whoever makes the comparison will easily agree with me, that this method is much readier and less embarrassed than either of their's.

LEMMA II.

A THEOREM.

180. Let there be two fractions $\frac{a}{b}$ and $\frac{c}{d}$ such, that if a continual division be made from a and b , according to the method for finding the greatest

Art. 180. A GENERAL DEFINITION OF PROPORTIONALITY. 289

greatest common measure, the quotients thence arising shall be found to be the same, both in quantity, order and number, with the quotients arising from a continual division made from c and d: I say then that

the fractions $\frac{a}{b}$ and $\frac{c}{d}$ will be equal one to the other.

To demonstrate this, let the quotients in both cases be 1, 2 and 3; and by the help of these quotients let a converging series of primary fractions be computed, as in the first scholium to the last article, to wit,

$\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{10}{7}$; then it is plain that this series will equally relate to both fractions, will equally converge to both, and at last terminate in

both; therefore the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ must be equal one to the other, since they are both equal to the last fraction of the series, to wit,

$\frac{10}{7}$. Q. E. D.

COROLLARY.

Since the two fractions mentioned in this article, to wit, $\frac{a}{b}$ and $\frac{c}{d}$ will be equal one to the other, whether the number of divisions and quotients be few or many, it is plain that the nature of this equality does not depend upon the supposition of a last division, though the demonstration does; and therefore this article will be true, even though the number of quotients be infinite.

SCHOLIUM.

If a, b, c and d , instead of numbers, represent any other quantities whatever, and the quotients from a and b be every way the same with the quotients from c and d , then the sense of the foregoing lemma is, that a will have the same proportion to b that c hath to d ; and that, whether the quantities a, b, c, d be commensurable or incommensurable: and therefore if we lay down this as a definition of proportionality, to wit, that *Four quantities are said to be proportionable, when the quotients of a continual division from the two first are every way the same with the quotients of a continual division from the two last*; this definition will reach incommensurables, and will sufficiently justify our application of proportion to such quantities: for now all the difference betwixt the proportion of commensurable and incommensurable quantities will only be this, that whereas in the case of commensurables, the quotients will break off, in the case of incommensurables they will run on

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ad infinitum; but the quotients from the two first will be every way the same with the quotients from the two last, which sufficiently marks out their proportionality, and is, as I take it, the most distinct definition that can be given of proportionality, though not so useful as that in the fifth definition to the fifth book of the elements, because the other properties of proportion are not so easily deducible from it.

OF INCOMMENSURABLES.

An apology for introducing the doctrine of Incommensurables into this place.

181. I doubt not but it will be thought by some a little out of method to introduce the doctrine of incommensurables into this place; but my first problem ought by no means to be rejected, whether we consider the elegance of the problem itself, or the close connexion it has with some of the foregoing articles, which it may serve further to illustrate; and as this problem relates purely to incommensurables, without a previous knowledge whereof it cannot possibly be understood, I thought this a proper place to bring my reader into some acquaintance with them; and if, after I found myself obliged to dip into this subject, I drew out from thence somewhat more than my present occasion required, it was because I easily foresaw that the rest would be of no less use on other occasions. In short, this doctrine of incommensurables is certainly very entertaining, and a curious instance of most subtil and refined reasoning, and therefore cannot, if closely and clearly delivered, but be acceptable to every ingenuous reader, who is really more inquisitive after true and substantial knowledge than after matter of criticism and censure; and this I hope will be sufficient to justify me.

I shall insert here but a very small part of what *Euclid* has delivered more at large upon this head; and my method will be, first to demonstrate some properties naturally flowing from the idea of incommensurability, and from one another, and then to find a subject for these properties, that is, to demonstrate that there are actually existing in nature such things as incommensurable quantities.

N. B. If any one has a mind to satisfy himself concerning the existence of incommensurables without being at the pains of reading all I have written upon the subject, he may first read the hundred eighty second article, and then pass over to the tenth proposition and those that follow, which are so contrived as to have little or no dependence upon the rest.

DEFI-

DEFINITIONS AND AXIOMS.

182. Def. 1st. One quantity is said to measure another, when being taken as often as possible out of that other, there is nothing left.

2d. When a quantity will measure two others, it is called their common measure.

3d. And quantities are said to be commensurable or incommensurable, according as they will or will not admit of a common measure.

4th. Whole numbers are said to be prime to one another, which will admit of no common measure but unity.

5th. A square number, in the sense we are here to use it, is any whole number that will admit of itself, or some other whole number for it's square root, exclusive of all fractions both out of the square and out of the root.

6th. Whenever a less whole number measures a greater, the less is called an aliquot part of the greater, and the greater a multiple of the less.

SCHOLIUM.

Fractions, as far as I can find, were very little regarded by the Ancients, especially before *Archimedes's* time; but what we now express and perform by fractions, they accomplished by the help of proportion in whole numbers; so that by number or numbers with them, must always be understood whole numbers, and by aliquot parts, such lesser whole numbers as will measure others that are greater: thus the aliquot parts of the number 6 were 1, 2 and 3; but the mixt number $1\frac{1}{2}$, or the fraction $\frac{3}{2}$, though it measures 6 just four times, was nevertheless excluded out of the number of it's aliquot parts: and to say the truth, if this be not done, every number will have as many aliquot parts as we please; and so the distinction of prime and composite numbers, and the distinction of numbers that are prime and not prime to each other, will be utterly lost; our idea of proportionality in numbers will be rendered obscure and confused; and that excellent treatise of numbers by *Euclid* in his seventh, eighth and ninth books, will become in a great measure useless. Not that I would here be understood as if I did not think fractions exceeding useful in all arithmetical operations; so far am I from thinking otherwise, that I have frequently used them myself in cases where others used proportion, purely because I looked upon fractions as more tractable: but what I would chiefly signify by the foregoing discourse is, that whether fractions be admitted into arithmetical operations or not, the doctrine of whole numbers being previous to, and distinct from that of fractions, the Ancients were very justifiable in treating it distinctly by itself, without any regard to fractions.

COROLLARIES FROM THE DEFINITIONS.

1st. *All whole numbers are commensurable*: for unity is a common measure to them all.

2d. *All fractions are commensurable*: for if the fractions $\frac{a}{b}$ and $\frac{c}{d}$ be reduced to the same denomination, they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$; and $\frac{1}{bd}$ will measure them both.

3d. *All whole numbers and mixt numbers compared together, all mixt numbers compared with one another, all whole numbers and fractions, all mixt numbers and fractions are commensurable*: for all mixt numbers may be reduced to fractions; and all whole numbers may be considered as fractions whose denominators are units.

Axiom 1. *If one quantity measures a second, and that a third, the first quantity will also measure the third.*

2d. *Whatever quantity measures two others, the same will measure both their sum and their difference.*

3d. *Two finite homogeneous quantities can never be so different one from another, but that the lesser may be multiplied till it exceeds the greater.*

4th. *Equal fractions may be converted into equal proportions in whole numbers, by making the numerators antecedents, and the denominators consequents; and vice versa, equal proportionals in whole numbers may be changed into equal fractions.*

PROPOSITION I.

183. *All commensurable quantities are to one another as number to number; and vice versa, all quantities that are to one another as number to number are commensurable.*

For first, let a and b be two commensurable quantities; I say then that the proportion of a to b , let it be what it will, may be expressed in whole numbers: for as a and b are commensurable quantities, let c measure them both; let c measure a just three times, and b four times; then will $3c = a$, and $4c = b$, and a will be to b as 3 to 4. Q. E. D.

Again, let a and b be two such quantities that the proportion of a to b may be expressed in whole numbers; as for example, let a be to b as 3 to 4, and let c be a third part of a ; then will $3c = a$, and $4c = b$, and so the quantity c will measure both a and b ; therefore the quantities a and b will be commensurable. Q. E. D.

PROPO-

PROPOSITION 2.

184. *Incommensurable quantities are not to each other as number to number; and vice versa, quantities that are not to each other as number to number are incommensurable.*

For first, let a and b be two incommensurable quantities; I say then that the proportion of a to b cannot possibly be expressed in whole numbers: for if it could, those quantities a and b would be commensurable, by the latter part of the foregoing proposition, which contradicts the supposition. In the next place, let the quantities a and b be such, that the proportion of a to b cannot possibly be expressed in whole numbers; I say then that those quantities a and b will be incommensurable: for if they were commensurable it might, by the former part of the foregoing proposition. Q. E. D.

COROLLARY.

If the proportion of incommensurable quantities be too subtil to be expressed in whole numbers, neither can it in fractions: for all fractions are commensurable, by the second corollary to the definitions, and as such, are to one another as number to number, by the first proposition.

PROPOSITION 3.

185. *If four quantities a , b , c and d be proportionable, so as that a is to b as c to d ; I say then that if the two first a and b be commensurable, the two last c and d will also be commensurable; but if the two first a and b be incommensurable, the two last c and d will also be incommensurable.*

For first, let a and b be commensurable; then will a be to b as number to number, by the first proposition; but as a is to b so is c to d ; therefore c will be to d as number to number; therefore c and d will be commensurable by the first proposition. Q. E. D.

In the next place, let a and b be incommensurable; I say then that c and d will also be incommensurable: for if c and d were commensurable, a and b would also be commensurable, by the former part of the proposition, because c is to d as a to b ; but a and b are not commensurable, by the supposition; therefore neither are c and d ; therefore c and d are incommensurable. Q. E. D.

COROLLARY.

By the demonstration of this proposition it appears, that *If incommensurability can be proved to belong to any one sort of continued quantity whatever,*

whatever, it will equally belong to all; for in this demonstration, a and b may be quantities of any one sort, and c and d those of another. The several sorts of continued quantity, such as length, superficial and solid space, time, weight, motion &c, are all equally subtil in their constitution; and therefore if a , b , c and d be all continued quantities, there can be no absurdity in supposing them all proportionable: but if the antecedent and consequent of one proportion be continued quantities, and those of the other discontinued or numbers, these latter being of a less subtil composition, cannot always be proportionable to the former, as appears from the second proposition; though on the other hand, the former may always be proportionable to the latter.

S C H O L I U M.

Hence appears the necessity of *Euclid's* treating proportionable magnitudes or continued quantities, which have no such thing as a least finite part, after a very different manner from proportionable numbers, whose least finite part is unity.

P R O P O S I T I O N 4.

186. *If two quantities be commensurable to a third, they will be commensurable to one another.*

Let a and b be two quantities commensurable to a third, which we will call c ; I say then that the quantities a and b will be commensurable to one another: for first, since a and c are commensurable, they must be to one another as number to number, by the first proposition: let then a be to c as the number d to the number e . Again, since b and c are supposed commensurable, let c be to b as the number f to the number g ; then since a is to c as d to e , or as df to ef ; and again, since c is to b as f to g , or as ef to eg , it follows *ex æquo*, that a is to b as the number df to the number eg , and consequently that a and b are commensurable, by the first proposition. Q. E. D.

C O R O L L A R Y 1.

Hence it appears that two incommensurable quantities cannot both be commensurable to a third; because none but commensurables can, by this proposition.

C O R O L L A R Y 2.

If there be three homogeneous quantities a , b and c , whereof a is incommensurable to b , and b is commensurable to c ; I say then that a is incommensurable to c : for if it was not, we should have two incommensurable

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 rable quantities a and b both commensurable to a third c , which, by the
 corollary foregoing, is impossible.

PROPOSITION 5.

187. *As incommensurable quantities cannot be equal one to another, so
 neither can any multiple, part or parts of one be equal to any multi-
 ple, part or parts of the other.*

That two incommensurable quantities, suppose a and b , cannot be equal is plain; for then either would measure both, and so they would not be incommensurable: and in the next place, that no multiple, part or parts of one can be equal to any multiple, part or parts of the other, I thus demonstrate: Let c be any multiple, part or parts of a , and d of b ; then if c be a multiple of a , it is plain that a will measure c ; on the other hand, if c be a part of a , c will measure a ; and lastly, if c be parts of a , as $\frac{2}{3}$ of a , then it is plain that $\frac{3}{2}$ part of a will measure them both; so that in all cases a will be commensurable to c ; and b to d , for the same reason; therefore c and d cannot be commensurable; for if they were, since a is commensurable to c , and c to d , a and d would be commensurable, by the last proposition; but if a be commensurable to d , and d to b , then a and b will be commensurable, by the same; but a and b are not commensurable; therefore neither will c and d . So far therefore are c and d from being equal, that they are not so much as commensurable to one another; therefore of two incommensurable quantities, no multiple, part or parts of one can be equal to any multiple, part or parts of the other. Q. E. D.

Otherwise more directly thus: a is incommensurable to b , and b is commensurable to d ; therefore d is incommensurable to a , by the second corollary to the foregoing proposition: but if d be incommensurable to a , and a be commensurable to c , then d will be incommensurable to c , by the same corollary; therefore &c. Q. E. D.

LEMMA.

188. *If of two homogeneous finite quantities a and q , from the greater a be taken more than it's half, and from the remainder more than it's half, and from the last remainder more than it's half, &c; I say then that this subtraction may be continued to a remainder less than q .*

For first, since by the supposition a and q are homogeneous and finite quantities, the quantity q cannot be so small but that it may be multiplied till it exceeds a , by the third axiom: let then $6q$ be a multiple of q that exceeds a ; then will $\frac{a}{6}$ be less than q : let now the half of a be taken

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ken from a , and then there will remain $\frac{1}{2}a$; take away again the half of this, and there will remain $\frac{1}{4}a$; take away again the half of this, and there will remain $\frac{1}{8}a$; but $\frac{1}{8}a$ is less than $\frac{1}{6}a$, and $\frac{1}{8}a$ is less than q ; therefore $\frac{1}{8}a$ is less than q ; therefore if from a be taken away it's half, and from the remainder it's half, and from the last remainder it's half, and so on, the subtraction may be continued to a remainder less than q ; therefore *a fortiori*, if from a be taken away more than it's half, and from the remainder more than it's half, and from the last remainder more than it's half, and so on, the subtraction may be continued to a remainder less than q . Q. E. D.

PROPOSITION. 6.

189. Let a and b be two homogeneous quantities; and from these two quantities let a continual division be formed in manner following: divide a by b , and let the remainder be c ; then divide b by c , and let the remainder be d ; then divide c by d , and let the remainder be e , &c: I say then that the series a, b, c, d, e &c may be continued to terms less than any assignable quantity as q .

a.	b.	c.	d.	e.	f.	g.	h.
41.	24.	17.	7.	3.	1.	0.	0.

For in the first division, when a was divided by b , that is, where b was taken away from a as often as possible, the remainder c must be less than b ; for if it was not so, b might have been taken out once more, or oftener, contrary to the supposition: since then the quantity taken away was b at least, if not $2b, 3b, 4b$ &c, it follows that the quantity taken away must be greater than that which was left; therefore in the first division, where a was divided by b , there was taken away from a more than it's half, and there remained c : in like manner in the third division, where c was divided by d , there was taken away from c more than it's half, and the remainder was e : so again in the fifth division there was taken away from e more than it's half, and the remainder was g , &c; therefore the terms a, c, e, g &c may be continued till they become less than q , by the foregoing lemma; in like manner, from the second, fourth and sixth divisions it appears, that the terms b, d, f, h may be continued till they become less than q : put both these considerations together, and then it will appear that the terms a, b, c, d, e, f, g, h may be continued till they become less than q . Q. E. D.

PROPOSITION. 7.

190. Supposing all things as in the last proposition; I say then that whatever quantity will measure any two terms of the foregoing series that lie next to one another, the same quantity will measure all the rest both before and after them.

Let

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Let q measure the two contiguous quantities d and e ; I say then in the first place, that q will measure all the rest that follow e : for when d was divided by e there remained the next term f ; therefore e measures $d-f$: since then q measures e *ex hypothesi*, and e measures $d-f$, q must measure $d-f$, by the first axiom; but q also measures d *ex hypothesi*; therefore q measures both the quantities d and $d-f$, whose difference is f ; therefore q measures f by the second axiom: if then because q measures d and e , it follows that it must necessarily measure f , so because q measures e and f , it must necessarily measure g , and so on. I say in the next place, that if q measures d and e , it will measure all the terms before d : for when c was divided by d there remained e ; therefore d measures $c-e$: since then q measures d *ex hypothesi*, and d measures $c-e$, q measures $c-e$; but q measures e also *ex hypothesi*; therefore q measures both e and $c-e$, whose sum is c ; therefore q measures c : since then upon a supposition that q measures both e and d , it follows that it must necessarily measure c , so because q measures both d and c , it must necessarily measure b , and so on that way. *Q. E. D.*

PROPOSITION 8.

191. *Supposing all things as in the two foregoing propositions, I say that if the original terms a and b be commensurable, the series a, b, c, d, e &c will not run on ad infinitum, but will break off, so that the last term of the series will be the greatest common measure of a and b ; and vice versa, if the terms a, b, c, d, e &c do not run on ad infinitum, I say then that the original terms a and b will be commensurable.*

For first, let a and b be commensurable; I say then that the series will not run on *ad infinitum*: for let q be any common measure of a and b , then will q measure all the subsequent terms by the last proposition; but if the terms run on *ad infinitum*, they will come at last to be less than q by the sixth proposition, and so q cannot measure them all; therefore the series does not run on *ad infinitum*: let then the last term be e , and I shall then prove in the next place, that e will be greatest common measure of a and b : for as e is the last term of the series by the supposition, it will precisely measure d the last but one without any remainder; for should any thing remain, as f , then f would be the next term after e , and so e would not be the last term of the series, contrary to the supposition: since then e measures both d and itself, it must also measure a and b by the last proposition; and I say further, that it will be the greatest quantity that can measure them both: for should a and b admit of a greater common measure than e , suppose q , then q would mea-

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sure

sure all the subsequent terms, and amongst the rest e , by the last proposition, that is, a greater quantity would measure a less, which is absurd; therefore if e be the last term of the series, it will be the greatest common measure a and b will admit of. *Q. E. D.*

I come in the last place to prove, that if the series a, b, c, d, e &c does not run on *ad infinitum*, the original quantities a and b will be commensurable: for since the series is supposed not to run on *ad infinitum*, let e be the last term; then will e measure both d and c as was proved before; therefore it will also measure a and b by the last proposition, and therefore a and b will be commensurable.

COROLLARY 1.

Hence if a and b be commensurable, whether they be numbers, or any other homogeneous quantities, it will be easy to find their greatest common measure; to wit, by computing the last term of the series a, b, c, d, e : thus the greatest common measure of the numbers 42 and 30 is 6, because 6 is the last term of the series 42, 30, 12, 6.

COROLLARY 2.

If the original quantities a and b be incommensurable, the series a, b, c, d, e &c will run on *ad infinitum*: for if it did not, a and b would be commensurable, by the latter part of this proposition.

COROLLARY 3.

And vice versa, if the series a, b, c, d, e &c runs on *ad infinitum*, the original quantities a and b will be incommensurable: for if they were commensurable, the series would not run on *ad infinitum*, by the former part of this proposition.

COROLLARY 4.

Whatever quantity, as q , will measure two others a and b , the same will also measure their greatest common measure e .

PROPOSITION 9.

A PROBLEM.

192. It is required, having given three commensurable quantities a, b and c , to find their greatest common measure.

SOLUTION.

First find the greatest common measure of any two of them, suppose of a and b , and call it d ; then again find the greatest common measure of

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of d and of the third quantity c , and call it e ; then will e be the greatest common measure of all the three quantities a , b and c first proposed. For first, since e measures d , and d measures both a and b , e will measure both a and b ; but e measures c *ex hypothesi*, being supposed the greatest common measure of d and c ; therefore e will measure all the three quantities a , b and c . I say in the next place, that e is the greatest quantity that can measure them all: for if it be possible, let f , a greater quantity than e , measure all the three quantities a , b and c ; then since f measures both a and b , it will also measure their greatest common measure d , by the fourth corollary to the last proposition; but f measures c *ex hypothesi*; therefore f will measure e , the greatest common measure of d and c , that is, a greater quantity will measure a less, which is absurd; therefore e is the greatest common measure of a , b and c . Q. E. D.

Of the foregoing rule take the following example in numbers; let it be proposed to find the greatest common measure of the numbers 42, 30 and 15; now the greatest common measure of 42 and 30 is 6, by the first corollary to the eighth proposition; and the greatest common measure of 6 and 15 is 3; therefore the greatest common measure of 42, 30 and 15 is 3; and if the numbers 42, 30 and 15 be all divided by 3, they will be reduced to 14, 10 and 5, three other numbers in the same proportion with the dividends respectively.

COROLLARY.

Whatsoever quantity, as f , will measure three quantities a , b and c , the same will measure their greatest common measure e .

N. B. The following propositions are to prove the existence of incommensurables, and some other affections of them, proper to be known.

PROPOSITION 10.

193. *All equal fractions, when reduced to their least terms, become one and the same fraction.*

Before I enter upon the demonstration of this proposition, it will be proper to take notice by way of lemma, that *If any two numbers, as 7 and 11, whose greatest common measure is unity, be made to multiply a third quantity, as d , of what kind soever, then will that quantity d be the greatest common measure of the quantities $7d$ and $11d$; and vice versa, if d be the greatest common measure of the quantities $7d$ and $11d$, then will unity be the greatest common measure of the numbers 7 and 11.* For if in finding the greatest common measure of 7 and 11, the divisors be 7, 4, 3, 1, as they actually are, then in finding the greatest common measure of $7d$ and $11d$, the divisors will be $7d$, $4d$, $3d$, $1d$; that is, if 1 be the last

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divisor,

divisor, and consequently the greatest common measure in the former case, d will be the last divisor, and consequently the greatest common measure in the latter case, and *vice versa*.

This being allowed, let there now be two equal fractions, which when reduced to their least terms, are $\frac{a}{b}$ and $\frac{c}{d}$: I am then to demonstrate, that the numbers c and d are actually the same with the numbers a and b respectively.

For first it is plain that the numbers a and b can have no common measure but unity, any more than the numbers c and d : for if they had, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ would not be in their least terms, but would be further divisible, contrary to the supposition.

2dly, The fractions $\frac{a}{b}$ and $\frac{c}{d}$ must be equal to each other: for by what methods soever these reductions to the least terms were made, this is certain, that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ must be equal to the fractions first proposed, otherwise they would not represent those fractions in their least terms, but other fractions: since then the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal to the fractions first proposed, and those fractions are equal to each other by the supposition, it follows that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are also equal.

3dly, Let now the equal fractions $\frac{a}{b}$ and $\frac{c}{d}$ be reduced to the same denomination, and the fractions thence arising, to wit, $\frac{ad}{bd}$ and $\frac{bc}{bd}$ will still be equal; because neither can this reduction affect their values, by the eighth article of the introduction: since then the fractions $\frac{ad}{bd}$ and $\frac{bc}{bd}$ are equal, and have the same denominator, they must also have the same numerator, that is, the number bc must be the same with the number ad ; therefore the numbers bc and bd must be the same with numbers ad and bd respectively; therefore the greatest common measure of the former couple will be the same with the greatest common measure of the latter; but the greatest common measure of the numbers bc and bd is b , by the lemma, because it has been proved that the greatest common measure of the numbers c and d is unity; and for a like reason, the greatest common measure of the numbers ad and bd is d ; therefore
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the numbers b and d are the same: since then the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, and have the same denominator, they must likewise have the same numerator, that is, the number c must be the same with the number a ; and so the numbers c and d must be actually the same with the numbers a and b respectively. Q. E. D.

COROLLARY.

If there be two equal fractions $\frac{a}{b}$ and $\frac{c}{d}$, whereof the former $\frac{a}{b}$ is in it's least terms; I say then that a the numerator of the former fraction will measure c the numerator of the latter, and that b the denominator of the former will measure d the denominator of the latter as often. For let e be the greatest common measure of the numbers c and d ; then will unity be the greatest common measure of the numbers $\frac{c}{e}$ and $\frac{d}{e}$ by the lemma laid down in the proposition; therefore $\frac{c}{e}$ and $\frac{d}{e}$ will be the least terms of the fraction $\frac{c}{d}$; but the fraction $\frac{c}{d}$ is equal to the fraction $\frac{a}{b}$, which is already in it's least terms by the supposition; and by this proposition the least terms of all equal fractions are the same; therefore the numbers $\frac{c}{e}$ and $\frac{d}{e}$ are the same with the numbers a and b ; therefore the numbers c and d are the same with the numbers ea and eb respectively; but a measures ea , and b measures eb as often; therefore a measures c , and b measures d as often. Q. E. D.

PROPOSITION II.

194. If there be three numbers a , b and c , whereof a is prime to b , and b is a multiple of c ; I say then that a will be prime to c .

For if a and c be not prime to each other, they must have some common measure greater than unity by the fourth definition; let that be d ; then since d measures c , and c measures it's multiple b , d will measure b ; but d also measures a , as being a common measure of a and c ; therefore d measures both a and b ; therefore a and b cannot be prime to each other; therefore if a and c be not prime to each other, neither can a and b ; but a and b are prime to each other by the supposition; therefore a and c must also be prime to each other. Q. E. D.

S C H O

SCHOLIUM.

When c is equal to unity, this proposition is self-evident; not so much that the numbers a and c are prime to each other, as that they will have no common measure but unity: should unity be admitted as a number, then every other number would be both prime to it, and a multiple of it; to avoid which confusion, and to establish a distinction betwixt numbers prime and not prime to each other, *Euclid* has excluded unity from his definition of number, chusing rather to consider it as a common denominator of all number: thus the number 3 is to be interpreted three units, &c.

PROPOSITION 12.

195. *If two numbers a and b be both prime to a third number c ; I say then that their product ab will also be prime to the same third number c ; or (which amounts to the same thing) that the product ab and the number c will have no common measure but unity.*

If this be denied, let the product ab and the number c have some common measure greater than unity, and let that be d ; then since d measures both the product ab and the number c , it will measure the number c alone, and consequently c will be some multiple of d ; therefore we have three numbers a , c and d , whereof a is prime to c , and c is a multiple of d ; therefore by the last proposition a is prime to d , and the fraction $\frac{a}{d}$ is in it's least terms: again, since d measures both the product ab and the number c , it will measure ab alone, and consequently ab will be some multiple of d ; therefore the fraction $\frac{ab}{d}$ will be equivalent to some whole number; call that whole number e , and then since $\frac{ab}{d} = e$, we shall have $\frac{a}{d} = \frac{e}{b}$; here then we have two equal fractions $\frac{a}{d}$ and $\frac{e}{b}$, whereof the former $\frac{a}{d}$ is in it's least terms; therefore d the denominator of the former fraction will measure b the denominator of the latter, by the corollary to the tenth proposition; but d measures c *ex hypothesi*; therefore d , a number greater than unity, measures both b and c ; therefore if the product ab and the number c be not prime to each other, neither can the numbers b and c ; but the numbers b and c are prime to each other *ex hypothesi*; therefore the product ab and the number c are also prime to each other. Q. E. D.

COROL.

COROLLARY 1.

If two numbers a and b be prime to other two c and d in such a manner that both the numbers a and b be prime to each of the numbers c and d ; I say then that ab the product of the former two will be prime to cd the product of the latter two. For since both a and b are prime to c , the product ab will also be prime to c by this twelfth proposition; and for the same reason the product ab will also be prime to the number d : since then both the numbers c and d are prime to the number ab , their product cd must also be prime to ab , and ab to cd . Q. E. D.

COROLLARY 2.

*If the numbers a and b in the foregoing corollary be supposed equal to one another, as also the numbers c and d ; and if moreover a be prime to c , then both a and b will be prime to each of the numbers c and d , and so the product ab will be prime to the product cd : but in this case, the product ab is the same as a^2 , and the product cd the same as c^2 ; therefore a^2 will be prime to c^2 ; which is as much as to say, that *If two numbers a and c be prime to each other, their squares aa and cc will be so too.**

COROLLARY 3.

Therefore if any two numbers a and c be the least in their proportion, their squares aa and cc will be the least in their's: for when either a or c is equal to unity, this is evident of itself (unity being incapable of any further division;) and when both a and c are greater than unity, aa and cc will be the least in their proportion by the last corollary.

PROPOSITION 13.

196. If there be three homogeneous quantities a , b and c in continual proportion, and if the middle term be commensurable to the extremes; I say then that the least numbers in the proportion of the extremes will be both squares.

Since a and b are supposed commensurable, let their greatest common measure be contained in a d times, and in b e times; then will d and e be the least numbers in the proportion of a to b , by the tenth proposition; and since a is to b as d is to e , or as dd to de , and since b is to c also as d to e , or as de to ee , it follows *ex æquo*, that a is to c as dd to ee ; but d and e are the least numbers in the proportion of a to b , as hath been already shewn; therefore dd and ee are the least numbers in the proportion of a to c , by the third corollary to the last proposition; therefore the least numbers in the proportion of the extremes a and c are both squares. Q. E. D.

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PROPOSITION 14.

197. *If there be three homogeneous quantities of any kind a, b and c in continual proportion, whose extremes a and c are commensurable to each other; and if the least numbers in the proportion of these extremes be not both squares; I say then that the middle quantity b will be incommensurable to both the extremes.*

That the middle quantity cannot be commensurable to one extreme, and incommensurable to the other, is evident; for if we should suppose a and b for instance, to be incommensurable, and at the same time b and c to be commensurable, we should then have two incommensurable quantities a and b both commensurable to the same third quantity c, which by the first corollary to the fourth proposition is impossible; therefore the middle quantity must either be commensurable to both the extremes or incommensurable; commensurable it cannot be, for then the least numbers in the proportion of the extremes must be both squares by the last proposition, which contradicts the *hypothesis*; therefore the middle term must be incommensurable to both the extremes. Q. E. D.

COROLLARY.

Hence having any line as a given, it will be easy to find as many others as we please, that shall all be incommensurable to it: for taking any two numbers d and e that are the least in their proportion, and not both squares, make c to a as d to e, and a mean proportional between a and c, found by the thirteenth of the sixth element, will be incommensurable to both.

PROPOSITION 15.

198. *If there be any whole number, as n, whose square root cannot be expressed by any other whole number; I say then that neither can it be expressed by any fraction whatever.*

For if possible, let the square root of n be expressed by a fraction which when reduced to it's least integral terms is $\frac{a}{b}$, that is, let $\frac{a}{b} = \sqrt{n}$; then we shall have $\frac{aa}{bb} = \frac{n}{1}$; but the fraction $\frac{aa}{bb}$ is in it's least terms, by the third corollary to the twelfth proposition, because the fraction $\frac{a}{b}$ was so; and the fraction $\frac{n}{1}$ is in it's least terms, because 1 cannot be further reduced; therefore we have two equal fractions $\frac{aa}{bb}$ and $\frac{n}{1}$, both in

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in their least terms; therefore by the tenth proposition, these two fractions must not only be equal in their values, but in their terms also, that is, aa must be equal to n , and bb to 1: but aa cannot be equal to n , because a is a whole number by the supposition, and n is supposed to admit of no whole number for it's root; therefore the square root of n cannot possibly be expressed by any fraction whatever. Q. E. D.

Otherwise thus: first, n , \sqrt{n} and 1 are continual proportionals, because the square of the middle term is equal to the product of the extremes; secondly, the extremes n and 1 are the least in their proportion, because 1 cannot be further reduced; thirdly, and one of the extremes n is no square number by the supposition; therefore the middle term which is \sqrt{n} is incommensurate to 1; therefore \sqrt{n} cannot possibly be expressed by any fraction or mixt number whatever, because these are all commensurate to 1 by the third corollary to the definitions. Q. E. D.

SCHOLIUM.

From this proposition it appears, that two surd numbers may be both incommensurable to unity, and yet both commensurable to one another: for $\sqrt{2}$ and $\sqrt{8}$ (by this proposition) are both incommensurable to unity, and yet they will both be commensurable to one another; for since 2 is to 8 as 1 is to 4, $\sqrt{2}$ will be to $\sqrt{8}$ as 1 to 2.

PROPOSITION 16.

199. If there be two numbers a and b the least in their proportion, and such whose square roots are commensurable one to the other; I say that the numbers a and b must be both squares.

For since \sqrt{a} is supposed commensurable to \sqrt{b} , let their greatest common measure be contained in \sqrt{a} , d times, and in \sqrt{b} , e times; then will d and e be the least numbers in the proportion of \sqrt{a} to \sqrt{b} by the tenth proposition; and since \sqrt{a} is to \sqrt{b} as d to e , we shall have a to b as dd to ee ; whence $\frac{a}{b} = \frac{dd}{ee}$ by the fourth axiom: but the

fraction $\frac{a}{b}$ is in it's least terms by the supposition, and so also is the fraction $\frac{dd}{ee}$, because the fraction $\frac{d}{e}$ was; therefore $a = dd$, and $b = ee$; therefore a and b are both squares. Q. E. D.

COROLLARY.

Therefore if a and b be the least in their proportion, and not both squares, as 1 and 2, 2 and 3, &c, their square roots will be incommensurable to one another.

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PROPOSITION 17.

200. *If two incommensurable quantities as 2 and $\sqrt{3}$ be put together, so as to constitute a third quantity $2 + \sqrt{3}$; I say then that the quantity so constituted shall be incommensurable to both the constituent parts.*

Thus I shall prove that $2 + \sqrt{3}$ is incommensurable to $\sqrt{3}$.

For should it be otherwise, both $2 + \sqrt{3}$ and $\sqrt{3}$ would have some common measure; let that be m ; and then since m measures both $2 + \sqrt{3}$ and $\sqrt{3}$, it will also measure their difference 2, and so m will measure both 2 and $\sqrt{3}$; but 2 and $\sqrt{3}$ are incommensurable by the supposition; therefore $2 + \sqrt{3}$ must not be commensurable to $\sqrt{3}$. Q. E. D. And the same demonstration may be applied, *mutatis mutandis*, to prove that $2 + \sqrt{3}$ is incommensurable to 2.

PROPOSITION 18.

201. *The side and diameter of a square are incommensurable.*

This follows from the fifteenth proposition; for by the fortyseventh of the first element, if the side of a square be called 1, its diameter will be $\sqrt{2}$; but 1 and $\sqrt{2}$ are incommensurable by the fifteenth proposition; therefore the side and diameter of a square are incommensurable. Q. E. D.

But this proposition may also be demonstrated independently of the foregoing propositions by the help of the following lemma.

LEMMA.

If a square number be even, not only it's root, but it's half will be so too.

For an odd root as $2a + 1$ produces an odd square, as $4aa + 4a + 1$; whereas an even root, as $2a$, produces not only an even square, as $4aa$, but it's half $2aa$ is likewise even.

This premised, if the side and diameter of any square be not incommensurable, they must have a common measure: let their greatest common measure be contained in the side a times, and in the diameter b times; then will a and b be the least numbers expressing the proportion of the side of a square to it's diameter: and since the side is to the diameter as a to b , the square of the side will be to the square of the diameter as aa to bb : but the square of the side is half the square of the diameter, by the fortyseventh of the first element; therefore aa is half of bb ; therefore aa and bb are both even numbers, bb for having an exact half, and aa for being that half, according to the lemma; therefore by the same

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same lemma, the roots of these even squares, to wit, a and b must both be even numbers; for if either of them was odd, its square must be so too: but now as these numbers are the least in their proportion, one of them at least must be an odd number, or the proportion would still be reducible to lower terms; therefore if the side and diameter of a square be not incommensurable, it will be possible for one and the same number to be both even and odd; but this is impossible; therefore they are incommensurable. Q. E. D.

Observations upon the whole.

202. *Scholium* 1. Thus we see how much more subtil the parts of continued quantities are than those of numbers; for though the root of 2 cannot possibly be expressed by any whole number or fraction, yet we see that if the side of any square be called 1, whether it be 1 yard, 1 foot, 1 inch, or whatever it is, the diameter will be the root of 2 upon the same scale; nay it will be very easy to construct a scale exhibiting the square roots of as many numbers as we please in their natural order from unity: but of this more in another place; see art. 307, scholium.

By this proposition we are also given to understand, that the areas of two squares may be commensurable, and at the same time the sides be incommensurable; for the area of any square is to the area of another square upon the diameter of the former as 1 to 2; and yet according to the last proposition, the sides of these two squares are incommensurable.

Scholium 2. In the corollary to the third proposition it is demonstrated, that if incommensurability can be proved to belong to any one sort of continued quantity whatever, it must equally belong to any other; but in the last proposition and in the corollary to the fourteenth it is demonstrated, that lines may be incommensurable; therefore all other continued quantities may. I shall only produce one instance of incommensurability in any other sort of quantity, and that shall be in time.

Monfieur *Huygens*, in his admirable treatise of the motion of pendulums, has demonstrated that the lengths of all pendulums, reckoning from the center of suspension to the center of oscillation, are as the squares of the times wherein they perform similar oscillations respectively: as if the length of one pendulum be three philosophical feet, and that of another two such feet, the square of the time of an oscillation of the former pendulum will be to the square of the time of a similar oscillation of the latter as 3 to 2; whence the times themselves will be as $\sqrt{3}$ to $\sqrt{2}$, and consequently may be represented by these two surds: let us now suppose what is pretty near the case, to wit, that $\sqrt{3}$ is to $\sqrt{2}$ as 49 to 40; then by multiplying extremes and means we shall have $\sqrt{3} \times 40 = \sqrt{2}$

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× 49,

$\times 49$, which is as much as to say that forty oscillations of the longer pendulum are performed in the same time as forty-nine of the shorter: now if this was exactly true, and if two clocks furnished with these pendulums should happen at any particular instant of time to beat together, after 40 beats caused by the longer pendulum, or 49 by the shorter, this coincidence would return again, and so on *ad infinitum*: this is upon a supposition that $\sqrt{3}$ is to $\sqrt{2}$ as 49 to 40, and consequently that $\sqrt{3} \times 40 = \sqrt{2} \times 49$; but if we suppose these two surds, and consequently the times they represent, to be (as they truly are) incommensurable, no multiple of one can be equal to any multiple of the other by the fifth proposition; and then it will follow that if at any particular instant of time these two clocks should happen to beat together, there can never happen such another instant, though their motion should continue to all eternity, provided the force of the wheels have no other influence upon these motions than to prevent retardations arising from other causes: they will frequently beat so close together that it will not be in the power of the ear to distinguish the difference; but another true mathematical coincidence can never happen after the first.

Scholium 3. Before I take my leave of this subject, I shall only further observe, that this doctrine of incommensurables quite overturns the *hypothesis* of indivisibles: for were there any such things as indivisibles, or least parts of magnitude, these would measure all the rest, since they could leave nothing less than themselves, and so there would be no such things as incommensurables, the existence of which we have already demonstrated beyond all contradiction.

PROBLEM I.

203. Let a and b be two incommensurable quantities, a a greater and b a less; and supposing a continual division to be made from a and b , according to the method for finding the greatest common measure, let the quotients thence arising be certain numbers always returning in the same order *ad infinitum*: It is required to determine the value of the fraction $\frac{a}{b}$, or (which is the same thing) to determine the proportion of a to b without any approximation, admitting surd numbers into the expression.

N. B. To determine this proportion by an approximation at pleasure, see scholium 1 to art. 179.

SOLUTION.

Let the returning quotients be 1, 2, 3. 1, 2, 3. 1, 2, 3. &c *ad infinitum*, and let the remainders of the first, second and third divisions be c, d

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c , d and e respectively; then will the quotient of a divided by b be 1, that of b divided by c , 2, that of c divided by d , 3; after which the quotient of d divided by e will again be 1, and so on: therefore a continual division begun from d and e and carried on *ad infinitum*, will be attended with the same quotients both in quantity, order and number as a continual division begun from a and b ; therefore d will have the same proportion to e as a to b , by art. 180. This being allowed, let a series of equations for a and b be computed as in the 175th article, and they will be as follows:

$$\begin{aligned} 1a - 0b &= +a. \\ 0a - 1b &= -b. \\ 1a - 1b &= +c. \\ 2a - 3b &= -d. \\ 7a - 10b &= +e. \text{ \&c.} \end{aligned}$$

Here then $d = 3b - 2a$, and $e = 7a - 10b$; but it has been proved above that d is to e as a to b ; therefore $3b - 2a$ is to $7a - 10b$ as a to b ; therefore by multiplying extremes and means, we have $3bb - 2ab = 7aa - 10ab$, or $7aa - 8ab = 3bb$: here then we have but one equation $7aa - 8ab = 3bb$ for determining two unknown quantities a and b , and therefore are at liberty to substitute what we please for one of them: let us then substitute such a quantity for b as will render the equation more simple thus; let $b = 7$, because 7 is the coefficient of aa in the equation; and then dividing the whole equation by 7, or it's equal b , we shall have $aa - 8a = 3b$, or $aa - 8a = 21$; this is a quadratic equation, and by completing the square we shall have $aa - 8a + 16 = 37$; whence by extracting the square root we have $a - 4 = \sqrt{37}$, and $a = 4 + \sqrt{37}$; therefore $\frac{a}{b} = \frac{4 + \sqrt{37}}{7}$, or (which is the same thing) a is to b as $4 + \sqrt{37}$ is to 7. Q. E. I.

N. B. 1st. Though the number 37, like all others, has two roots, yet I chuse the affirmative one, because $4 - \sqrt{37}$ is a negative quantity.

2^{dly}. If you would express the numerator of the fraction $\frac{a}{b}$ by a whole number, and the denominator by a mixt surd, in the foregoing equation $7aa - 8ab = 3bb$ you must make $a = 3$ the coefficient of bb , and then dividing the whole equation by a or 3, you will have $7a - 8b = bb$, that is, $bb + 8b = 21$, which being resolved gives $b = \sqrt{37} - 4$; whence $\frac{a}{b} = \frac{3}{\sqrt{37} - 4}$, or a is to b as 3 is to $\sqrt{37} - 4$.

3^{dly}. To confirm these two proportions, to wit, that a is to b as $\sqrt{37} + 4$ is to 7, and also that a is to b as 3 is to $\sqrt{37} - 4$, we are to

to take notice that the square root of 37 is 6.08276253 nearly, and consequently that $\sqrt{37+4}$ is to 7 as 10.08276253 is to 7, or as 1008276253 is to 700000000. Now if a continual division be made from the numbers 1008276253 and 700000000, the quotients will be found to be 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, the law of the series now breaking off, because the square root of 37 here given was not exact: on the other hand, that a is to b also as 3 to $\sqrt{37}-4$, that is, as 300000000 is to 208276253, will be evident by forming a continual division from the numbers 300000000 and 208276253, where the quotients will be found to be 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, &c.

4thly. From the solution of the foregoing problem it appears, that the square root of 37 cannot possibly be expressed by any whole number, mixt number, or fraction of what kind soever; for if it could, then $4+\sqrt{37}$ might be expressed so too, and consequently would not be incommensurable to the whole number 7, as by the infinity of quotients we find it is: and the same may be observed of all other surds found by the method of this solution.

5thly. If we would form a theorem for a general solution of the foregoing problem, we must look back a little upon the particular one; and considering it more attentively, we shall find a method of solving it in general thus; after a series of equations was formed according to art. 175 by the help of one circulation of the quotients 1, 2, 3, we found the two last equations to be $2a-3b=-d$, and $7a-10b=+e$; whence I infer, that if (according to art. 179) a series of primary fractions be computed by the help of the quotients 1, 2, 3, the two last fractions in the series will be $\frac{2}{3}$ and $\frac{10}{7}$: let us now examine the last equation from which the quantity a was found, to wit, $aa-8a=21$ or $3b$, and upon reflection we shall find that 8 the coefficient of $-a$ in the second term was found by subtracting 2 from 10, that is, by subtracting the denominator of the last fraction but one from the numerator of the last: we shall find also that 7 the value of b came out from 7 the denominator of the last fraction: lastly we shall find that 3 the coefficient of b in the term $3b$, or of 7 in the number 21 came from 3 the numerator of the last fraction but one, and consequently that $3b$ or 21 was the product of 3 the numerator of the last fraction but one multiplied into b or 7 the denominator of the last: all which considerations put together furnish us with the following general solution.

By the help of the first circulation of quotients compute (as in article one-hundred seventynine) a series of primary fractions; which done, multiply the numerator of the last fraction but one into the denominator of the last, and call the product p; subtract the denominator of the last fraction but one from
the

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the numerator of the last, and call the difference d; then you will have b equal to the denominator of the last fraction, and $aa - da = p$.

EXAMPLE 1.

Let the recurring quotients be 4, 1, 1, 1, 4, 1, 1, 1. &c *ad infinitum*: now a series of primary fractions computed by the means of the quotients 4, 1, 1, 1 is $\frac{0}{1}, \frac{1}{0}, \frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{14}{3}$; and 9 the numerator of the last fraction but one multiplied into 3 the denominator of the last gives $27 = p$; and moreover 2 the denominator of the last fraction but one subtracted from 14 the numerator of the last leaves $12 = d$: make therefore $b = 3$ the denominator of the last fraction, and you will have $aa - 12a = 27$; which equation resolved gives $a = 6 + \sqrt{63}$; therefore $\frac{a}{b} = \frac{6 + \sqrt{63}}{3}$; but this fraction may be reduced lower; for $\frac{6}{3} = \frac{2}{1}$, and $\frac{\sqrt{63}}{3} = \frac{\sqrt{63}}{3}$; therefore $\frac{a}{b} = \frac{2 + \sqrt{7}}{1}$.

EXAMPLE 2.

Let the quotients be q, q, q, q, q *ad infinitum*; then a series of fractions computed by the quotient q will be $\frac{0}{1}, \frac{1}{0}, \frac{q}{1}$, where the two last fractions are $\frac{1}{0}$ and $\frac{q}{1}$: now 1 the numerator of the last fraction but one multiplied into 1 the denominator of the last gives $1 = p$; and 0 the denominator of the last fraction but one subtracted from q the numerator of the last leaves $q = d$; make therefore $b = 1$ the denominator of the last fraction, and you will have $aa - qa = 1$: let $q = 2$, and the equation will be $aa - 2a = 1$; whence $a = 1 + \sqrt{2}$, and $\frac{a}{b} = \frac{1 + \sqrt{2}}{1}$; therefore if a continual division be made from the quantities $1 + \sqrt{2}$ and 1, the quotients will be 2, 2, 2, 2, 2 *ad infinitum*; but if $\sqrt{2}$ be divided by 1, the divisor and the remainder will be the same as when $1 + \sqrt{2}$ was divided by 1; therefore if $\sqrt{2}$ be divided by 1, all the quotients after the first will be the same as when $1 + \sqrt{2}$ was divided by 1; but the first quotient in the former case will be 1, and not 2 as in the latter; therefore if a continual division be formed from the quantities $\sqrt{2}$ and 1, the first quotient will be 1, and the rest will be 2, 2, 2, 2, 2 *ad infinitum*: now every one that knows any thing of Geometry, knows that

that the diagonal is to the side of any square as $\sqrt{2}$ to 1; whence it follows from the fourth observation of this article, that the diagonal and side of a square are incommensurable, and that if a continual division be formed from them, the first quotient will be 1, and the rest will be 2, 2, 2, 2, 2 *ad infinitum*. Again, let $q=1$, and we shall have $b=1$ as before, and $aa-a=1$: let the greater segment of a line cut in extreme and mean proportion be to the less as a to 1; then since (according to the nature of this section) the whole line is to the greater segment as the greater segment is to the less, we have this proportion, $a+1$ is to a as a is to 1; and this equation, $aa=a+1$, and $aa-a=1$: since then we fall into the same equation, whether we suppose $q=1$, or suppose a to be to 1 or b as the greater segment of a line divided in extreme and mean proportion is to the less, it follows that a is to b in that proportion; and consequently that if a continual division be formed from those segments, the quotients will be 1, 1, 1, 1, 1 *ad infinitum*.

EXAMPLE 3.

Let the circulating quotients be $q, r, q, r, q, r, \&c$ *ad infinitum*: here a series of primary fractions computed from the quotients q and r will be $\frac{0}{1}, \frac{1}{0}, \frac{q}{1}, \frac{qr+1}{r}$; where q the numerator of the last fraction but one multiplied into r the denominator of the last gives $qr=p$; and 1 the denominator of the last fraction but one subtracted from $qr+1$ the numerator of the last leaves also $qr=d$; make therefore $b=r$ the denominator of the last fraction, and a will be determined by this equation, to wit, $aa-qr a=qr$: if $q=2$, and $r=1$, we shall have a to b as $1+\sqrt{3}$ is to 1; therefore in a continual division begun from $1+\sqrt{3}$ and 1, the quotients will be 2, 1, 2, 1, 2, 1, *&c ad infinitum*; but in a continual division begun from $\sqrt{3}$ and 1, the first quotient will be 1, and the rest will be 1, 2, 1, 2, 1, 2, *&c ad infinitum*. If $q=1$, and $r=3$, a will be to b as $\frac{3+\sqrt{21}}{2}$ is to 3, or as $3+\sqrt{21}$ to 6.

If some of the first quotients are intended to be irregular, and the rest to return in a circle *ad infinitum*, proceed as follows: Let it be required that n and p shall be the two first quotients, and that the rest shall be $q, r, s, q, r, s, q, r, s, \&c$ *ad infinitum*: first then find by the foregoing problem two quantities a and b that shall have the regular quotients above described *ad infinitum* without any others; then make $pa+b=B$, and $nB+a=A$, and the two quantities A and B will have the quotients required. For since $A=nB+a$, it follows that if A be divided by B , the quotient will be n , and the remainder will be a : in like manner, since $B=pa+b$, it follows

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follows that if B be divided by a , the quotient will be p , and the remainder will be b : the next division is that of a by b , and so on; and therefore if this division be continued *ad infinitum*, the rest of the quotients will be $q, r, s. q, r, s. q, r, s. \&c$ *ad infinitum* by the supposition.

PROBLEM 2.

204. *I owe a friend a shilling, or some number of shillings, and we have nothing but guineas and louis'd'ors about us; the guineas being valued at twentyone shillings apiece, and the louis'd'ors at seventeen: The question is, How must I acquit myself of this debt?*

SOLUTION.

Since by the supposition this debt cannot otherwise be discharged than either by my paying guineas and receiving louis'd'ors, or else paying louis'd'ors and receiving guineas, or perhaps paying both guineas and louis'd'ors, it follows that this debt must be either the sum or the difference of a certain number of guineas and a certain number of louis'd'ors, and consequently that it must either be one shilling, which is the greatest common measure of a guinea and a louis'd'or, or else some number of shillings without a fraction; for nothing is more evident than that whatever measures any two quantities, must measure not only their multiples, but also the sums and differences of those multiples, as hath been before observed: and this is all the limitation this problem is subject to. This being then laid down, I make $a = 21$, and $b = 17$, and from these two values of a and b I derive a set of equations as in art. 175, *viz.*

$$\begin{aligned} 1a - 0b &= +21. \\ 0a - 1b &= -17. \\ a - b &= +4. \\ 4a - 5b &= -1. \\ 13a - 16b &= +1. \end{aligned}$$

Of these equations the two last will solve the problem as follows.

CASE I.

Let the debt be one shilling: then if I would make my payment in guineas and receive louis'd'ors, of the two last equations I make choice of that whose absolute term is affirmative, to wit, $13a - 16b = +1$, whereby it manifestly appears that if I pay my friend thirteen guineas, and he returns back again to me sixteen louis'd'ors, the debt will be discharged; for paying thirteen guineas and receiving sixteen louis'd'ors is equivalent to paying thirteen guineas *minus* sixteen louis'd'ors, which ac-

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cording

cording to the equation above quoted amounts to just one shilling: on the other hand, if I would make my payment in louis'd'ors and receive guineas, of the two last equations I must have recourse to that whose absolute term is negative, to wit, $4a - 5b = -1$, or $5b - 4a = +1$, whereby it appears that if I pay five louis'd'ors and receive back four guineas, I shall have discharged the debt: and as these equations $4a - 5b = -1$, and $13a - 16b = +1$ are the simplest of their kind by art. 177, the solutions deduced from them must be so too.

Now to find other solutions of this problem; since a is to b as 21 to 17, we have (by multiplying extremes and means) $17a = 21b$, or $17a - 21b = 0$; and as the fraction $\frac{21}{17}$ is in it's least terms, the equation

$17a - 21b = 0$ is the simplest of it's kind: add now the equation $17a - 21b = 0$ to the equation $13a - 16b = 1$ as often as you please by a continual addition, and also the equation $21b - 17a = 0$ to the equation $5b - 4a = 1$ in like manner, and you will have an infinite number of solutions to this problem, whether the payment is to be made in guineas and the balance in louis'd'ors, or the payment in louis'd'ors and the balance in guineas.

CASE 2.

Let now the debt be five shillings: then if I would pay guineas and receive louis'd'ors, I multiply the equation $13a - 16b = 1$ by 5 the number expressing the debt, and so have $65a - 80b = 5$: now to find the simplest equation of this kind, that is, whose absolute term is $+5$, I subtract, as often as I can, the equation $17a - 21b = 0$ from the equation $65a - 80b = 5$; and to do this the shortest way, I divide $65a$ by $17a$, that is, 65 by 17, and find the quotient to be 3 with a remainder; whence I infer that I can take the equation $17a - 21b = 0$ three times out of the equation $65a - 80b = 5$, and yet leave the coefficient of a affirmative: now $17a - 21b = 0$ being multiplied by 3 gives $51a - 63b = 0$, and this subtracted from $65a - 80b = 5$ gives $14a - 17b = 5$; therefore the simplest way of discharging the debt by making payment in guineas, is to pay 14 guineas and to receive 17 louis'd'ors: and if from this last equation $14a - 17b = 5$ again be subtracted $17a - 21b = 0$, there will remain $4b - 3a = 5$, which gives the simplest way of making the payment in louis'd'ors.

CASE 3.

Lastly let the debt be 100 pounds or 2000 shillings: then to solve this case, multiply the equation $13a - 16b = 1$ by 2000, and you will have $26000a - 32000b = 2000$: for a more simple equation, divide

26000

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26000 by 17, to wit, the coefficient of a in the equation $26000a - 32000b = 2000$ by the coefficient of a in the equation $17a - 21b = 0$, and the quotient will be 1529 with a remainder: subtract therefore 1529 times the equation $17a - 21b = 0$, that is, the equation $25993a - 32109b = 0$ from the equation $26000a - 32000b = 2000$, and you will have the equation $7a + 109b = 2000$; which shews that if I pay my friend 7 guineas and 109 louis'd'ors, I shall discharge the debt of one hundred pounds without receiving any thing back by way of exchange.

This then is one way of making up one hundred pounds out of guineas and louis'd'ors; and as many other ways may be found as the equation $17a - 21b = 0$ can be added to the equation $7a + 109b = 2000$, without absolutely destroying the coefficient of b , or reducing it to a negative: to know therefore how often this can be done, divide 109 the coefficient of b in the equation $7a + 109b = 2000$ by 21 the coefficient of $-b$ in the equation $17a - 21b = 0$, and the quotient will be 5 with a remainder; therefore there are five other ways of making up one hundred pounds out of guineas and louis'd'ors besides that already found, and consequently six ways in all, to wit,

7 guineas and 109 louis'd'ors,
 24 guineas and 88 louis'd'ors,
 41 guineas and 67 louis'd'ors,
 58 guineas and 46 louis'd'ors,
 75 guineas and 25 louis'd'ors, and
 92 guineas and 4 louis'd'ors.

The first of these ways solves the problem with the fewest guineas, and last with the fewest louis'd'ors possible, and therefore may be called the extreme solutions of the problem: but if the extreme solutions be required without the intermediates, then to one of the extreme equations already found, to wit, $7a + 109b = 2000$ add 5 times the equation $17a - 21b = 0$, to wit, the equation $85a - 105b = 0$, because there were five other solutions found, and you will have $92a + 4b = 2000$, which is the other extreme equation.

Besides the six ways of discharging the debt without interchanging, there are infinite other ways of doing it by way of interchange, as will appear by a continual addition and subtraction of the equation $17a - 21b = 0$. Q. E. I.

A DEFINITION.

205. Let q, r, s, t be any numbers given, and let a, b, c, d, e, f be a series whereof the two first terms a and b are known, and the rest are obtained in the following manner: make $qb + a = c$, $rc + b = d$, $sd + c = e$, $te + d = f$; then may the series a, b, c, d, e, f be said to be formed or generated

nerated from it's two first terms a and b by the numbers q, r, s, t . Thus a series generated from 1 and 0 by the numbers 2, 3, 4, 5, 6 will be 1, 0, 1, 3, 13, 68, 421; and a series generated from 0 and 1 by the numbers 3, 4, 5, 6 will be 0, 1, 3, 13, 68, 421: whence it appears, that If a series be formed from 1 and 0 by any set of numbers 2, 3, 4, 5, 6; and then setting aside the first number 2, another series be formed from 0 and 1 by the remaining numbers 3, 4, 5, 6, this latter series will be the same with the former, after the first term 1 is taken away.

PROBLEM 3.

206. It is required to find as many numbers as we please of this character, to wit, that if any one of them be divided by two given divisors, a a greater, and b a less, they shall have two given remainders d and e respectively.

SOLUTION.

Let g be any one of the numbers sought, and let l be the greatest common measure of the two given divisors a and b ; then since l measures a , and a measures $\overline{g-d}$ by the supposition, it follows that l must measure $\overline{g-d}$; in like manner, since l measures b , and b measures $\overline{g-e}$, l must measure $\overline{g-e}$; therefore l measures both the numbers $\overline{g-d}$ and $\overline{g-e}$; therefore if the former number be subtracted from the latter, l must measure the remainder $\overline{d-e}$; therefore if this problem be possible, $\frac{d-e}{l}$ must be reducible to a whole number, either affirmative or negative as d happens to be greater or less than e . This premised, let now the two values of a and b be particularly assigned; as for example, let $a=105$ and $b=40$, and let a series of equations be formed from these two values of a and b according to art. 175, as follows:

Equation	1,	$1a - 0b = + 105.$
	2,	$0a - 1b = - 40.$
	3,	$1a - 2b = + 25.$
	4,	$1a - 3b = - 15.$
	5,	$2a - 5b = + 10.$
	6,	$3a - 8b = - 5.$
	7,	$5a - 13b = + 5.$

From the two last of these equations it appears that 5 is the greatest common measure of a and b , (see art. 175;) and consequently that $\frac{d-e}{5}$ must be a whole number in all possible cases of this problem. Take now the

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the last equation of the series whose absolute term is negative, as $3a - 8b = -5$, and you will have $3a + 5 = 8b$; multiply the terms of this last equation into the number $\frac{d-e}{5}$, and you will have $3a \times \frac{d-e}{5} + 5 \times \frac{d-e}{5} = 8b \times \frac{d-e}{5}$, that is, $3a \times \frac{d-e}{5} + d - e = 8b \times \frac{d-e}{5}$; transpose $-e$, and you will have $3a \times \frac{d-e}{5} + d = 8b \times \frac{d-e}{5} + e$: but the number $3a \times \frac{d-e}{5}$ being divided by a , quotes $3 \times \frac{d-e}{5}$, and there remains nothing; therefore the number $3a \times \frac{d-e}{5} + d$ being divided by a , there will remain d ; in like manner if the number $8b \times \frac{d-e}{5} + e$ be divided by b , there will remain e : since then $3a \times \frac{d-e}{5} + d$ is equal to $8b \times \frac{d-e}{5} + e$, the same number $3a \times \frac{d-e}{5} + d$, or $\frac{3a}{5} \times d - e + d$ will answer both the conditions of the problem; but $a = 105$ *ex hypothesis*; therefore $\frac{3a}{5} \times d - e + d = 63 \times d - e + d$; therefore the number $63 \times d - e + d$ will answer both the conditions.

This one solution being obtained, innumerable others may be had by a continual addition and subtraction of 840 the least common multiple of a and b , which how to find, see art. 171: for as this number being severally divided by 105 and 40 will have no remainder, it follows that if this common multiple, or any number of them be added to or subtracted from $63 \times d - e + d$, the remainders, when the division is made by a and b , will be entirely the same as before that addition or subtraction, and therefore the remainders will still be d and e ; and this will give us an infinity of numbers with the properties above described. *Q. E. I.*

Thus have we a general theorem computed for the divisors 105 and 40, be the remainders d and e what they will, provided $\frac{d-e}{5}$ be a whole number: as for example, let it be required to find a number which being severally divided by 105 and 40, will have 39 and 9 for the respective remainders: here $d = 39$, $e = 9$, $d - e = 30$, $63 \times d - e = 1890$, $63 \times d - e + d = 1929$; therefore 1929 is such a number as will answer the conditions of the question: but if you would have the least number

number possible that will do the same thing, then cast away the number 840 (the least common multiple of 105 and 40) as often as possible, from 1929, that is, divide 1929 by 840, and the remainder 249 will be the least number which being divided by 105 and 40 will have 39 and 9 for their respective remainders. Again, let the remainders, instead of being 39 and 9, be 9 and 39 for a and b respectively; then will $d - e$ be -30 , and $63 \times \overline{d - e} = -1890$, and $63 \times \overline{d - e} + d = -1890 + 9 = -1881$; divide -1881 by 840, and the remainder is -201 , which is the least negative number of this kind; but we want the least affirmative number; and therefore if to the negative number already found, to wit, -201 be added the least common multiple 840, you will have 639 the least affirmative number which being divided by 105 and 40 will have 9 and 39 respectively remaining.

If any one be desirous of a synthetical demonstration to prove that the number $63 \times \overline{d - e} + d$, or $64d - 63e$ being severally divided by 105 and 40 will have d and e remaining respectively, he is to take notice, that if the divisors 105 and 40 had been prime to each other, this would easily have appeared by a simple division of the number $64d - 63e$ by 105 and 40; but as it is otherwise here, some further considerations are necessary, which are as follows. Since, when $64d - 63e$ is divided by 105, the last remainder is to be d , therefore I exterminate the other quantity $-63e$ thus: the greatest common measure of 105 and 40 is 5; therefore $\frac{d - e}{5}$ is a whole number, by what has been proved already;

now $63 \times \overline{d - e}$ is equal to $5 \times 63 \times \frac{d - e}{5}$, or $315 \times \frac{d - e}{5}$, both factors whereof are whole numbers; and $315 \times \frac{d - e}{5}$ is divisible by 105 without any remainder, because 315 is so; therefore $63d - 63e$ is divisible by 105 without any remainder; therefore if $63d - 63e$ be added to or subtracted from any number whatever, and the sum or the remainder be divided by 105, the remainder of this division will be the same as if such addition or subtraction had never been made: subtract then $63d - 63e$ from $64d - 63e$, and there remains d ; therefore $64d - 63e$ being divided by 105 will have d for a remainder. In the next place I divide $64d - 63e$ by 40, and there remains $24d - 23e$; but if the theorem be true, the number e ought to be the last remainder; to try which, I expunge the other quantity $24d$ out of the remainder $24d - 23e$ thus: $24 \times \overline{d - e}$, or $120 \times \frac{d - e}{5}$ is a number divisible by 40 with-

out

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out a remainder ; therefore $24d - 24e$ is divisible by 40 ; subtract then $24d - 24e$ from $24d - 23e$, and there remains e ; therefore $64d - 63e$ being divided by 40 will have e for a remainder. *Q. E. D.*

We come now to form a general theorem for any two divisors a and b whatsoever, where the remainders are possible ; and therefore must look back upon the foregoing solution, where it was found that the number

$\frac{3a}{5} \times \overline{d - e} + d$ would answer the conditions of the question. Now any one who observes how this solution was obtained, will easily perceive

that in this expression $\frac{3a}{5} \times \overline{d - e} + d$, the number 3 was the coefficient of a in the last equation of the foregoing series whose absolute term is negative, and that the number 5 was the greatest common measure of the two given divisors a and b : make then $r = 3$, $l = 5$ as before,

and the expression $\frac{3a}{5} \times \overline{d - e} + d$ will now be changed into this, $\frac{ra}{l} \times$

$\overline{d - e} + d$: but the greatest difficulty still remains behind, which is to trace out the number r by itself without being obliged to compute all the equations in the foregoing series, that is, to find out the coefficient of a in the last equation of those whose absolute terms are negative, or (which is the same thing) to find out the coefficient of a in the last equation that possesses an even place in the series ; for by a bare inspection of these serieses, every one may see that all the even places are taken up by equations whose absolute terms are negative. Now whoever considers with any degree of attention the nature and *genesis* of these equations, cannot but see that the coefficients of a , from the first equation to the last, are a series of numbers computed from 1 and 0 by the quotients of a continual division from the divisors a and b ; and from the last article it appears that such a series is always equivalent to a series begun from 0 and 1, and computed by the same quotients except the first ; therefore in finding the number r , the first of these quotients may always be set aside, provided that the computation be made to begin from 0 and 1. Again, if the number of quotients as they first come out be odd, the last quotient will lead us to a coefficient of a that stands in an odd place ; whereas we want the last coefficient in an even place ; therefore if the number of quotients be odd, both the first and last must be rejected ; but if the number be even, then only the first quotient must be rejected, and the last must be reduced, that is, diminished by unity, as is always done in computing the last equation, whether it happens in an even or an odd place, (see art. 175,) and the equation we are now enquiring after

ter will, in this case, be the last in the series. This known, if a series of numbers be computed from 0 and 1 by the help of the quotients still retained, the last term of the series will be r , and so the number $\frac{ra}{l} \times \overline{d-e} + d$ (which answers the conditions of the question) will be known.

If it be asked, what must be done when (in finding the greatest common measure l) the number of quotients is even, and the last is unity, since unity cannot be reduced; my answer is, that such a case can never happen.

EXAMPLE 1.

Let $a=105$, $b=40$, and consequently $l=5$, as in the foregoing problem; and the quotients of a continual division from a and b will be 2, 1, 1, 1, 2; reject the first and last, because of their odd number, and with the remaining quotients 1, 1, 1 form from 0 and 1 the following series 0, 1, 1, 2, 3, and r will be the last term 3; whence $\frac{ra}{l} \times$

$\overline{d-e} + d$, will be $\frac{3a}{5} \times \overline{d-e} + d = 63 \times \overline{d-e} + d$.

N. B. So long as the divisors a and b are in the same proportion one to the other, the expression of the number sought, to wit, $\frac{ra}{l} \times \overline{d-e} + d$ will be the same, let the divisors themselves be what they will; for the numbers $\frac{a}{l}$ and $\frac{b}{l}$ being the least in their proportion, must always be the same, and so will the quotients from which the number r is derived; all the difference then will only be the different restrictions to which the remainders d and e will be liable; for if this problem be possible, $\frac{d-e}{l}$ must always be a whole number, as was shewn before: thus if instead of making the divisors 105 and 40, we had made them 21 and 8, the least in their proportion, the number $63 \times \overline{d-e} + d$ would still have satisfied the conditions of the problem; and then the remainders d and e might have been made any two whole numbers whatever, since $\frac{d-e}{1}$ will always be a whole number.

EXAMPLE 2.

Let a and b the given divisors be 840 and 36, whose greatest common measure l is 12, and the quotients in finding it 23 and 3, even in number;

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number; drop therefore 23, and reduce the other quotient 3 to 2, and the series from 0 and 1 will be 0, 1, 2; therefore in this case $r=2$, and $\frac{ra}{l} \times \overline{d-e} + d$ will be $140 \times \overline{d-e} + d = 141d - 140e$, which is such a number as the problem requires, as may be thus demonstrated synthetically. First I exterminate $-140e$, by observing that $140d - 140e$, or $140 \times \overline{d-e}$, or $1680 \times \frac{d-e}{12}$ is a number divisible by 840; therefore $140d - 140e$ may be subtracted from $141d - 140e$ without affecting the remainder; subtract it then, and there remains d : next I exterminate d thus; $141d - 141e$, or $141 \times \overline{d-e}$, or $1692 \times \frac{d-e}{12}$ is a number divisible by 36, and therefore $141d - 141e$ may be subtracted from $141d - 140e$ without affecting the remainder; subtract it therefore and you will have e remaining.

EXAMPLE 3.

Make $a=9$, $b=7$, and consequently $l=1$, and the quotients in finding l will be 1, 3, 2, odd in number; drop the first and last, retaining only the middle quotient 3, and the series will be 0, 1, 3; therefore $r=3$, and $\frac{ra}{l} \times \overline{d-e} + d = 27 \times \overline{d-e} + d = 28d - 27e$; and as the divisors 9 and 7 are prime to each other, the synthetical demonstration will be very easy; for if $28d$ be divided by 9, the remainder will be d , and if $-27e$ be divided by 9, the remainder will be 0; therefore $28d - 27e$ divided by 9 will have d remaining: again, if $28d$ be divided by 7, the remainder will be nothing, and if $-27e$, that is, if $-28e + e$ be divided by 7, the remainder will be e ; therefore $28d - 27e$ divided by 7 will have e remaining: and this will always be the case when the divisors d and e are prime to each other; that is, the truth of the canon may be shewn by a simple division without any other artifice, the remainders in this case being subjected to no restriction.

EXAMPLE 4.

Let $a=30$, $b=10$, and consequently $l=10$, and there will be but one quotient of a continual division from 30 and 10, to wit 3; and this 3 must be rejected upon a double account, not only as it is the first quotient, but also as it is the last, since the number of quotients is an odd number; therefore to the two initial terms 0 and 1 of the series no other terms can be added, because there are no quotients whereby any other terms

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can be generated; therefore in this case, the latter term 1 must be looked upon as the last term of the series. Here then $r=1$, and $\frac{ra}{l} \times \overline{d-e} + d = 3 \times \overline{d-e} + d = 4d - 3e$, which I thus further demonstrate: $3d - 3e$, or $30 \times \frac{d-e}{10}$ is a multiple of 30; subtract then $3d - 3e$ from $4d - 3e$, and there remains d : again, $4d - 4e$, or $40 \times \frac{d-e}{10}$ is a multiple of 10; subtract then $4d - 4e$ from $4d - 3e$, and there remains e .

EXAMPLE 5.

Let $a=13$, and $b=7$, and consequently $l=1$; moreover let $e=0$; and the quotients of a continual division from 13 and 7 will be 1, 1, 6; drop the first and last on account of their odd number, and the remaining quotient will be 1, and the series will be 0, 1, 1; therefore $r=1$, and $\frac{ra}{l} \times \overline{d-e} + d = 14d$, which will answer the conditions; for $14d$ divided by 13 will have d remaining, and $14d$ divided by 7 will have nothing remaining.

EXAMPLE 6.

Suppose the present year of Christ to be one thousand seven hundred thirty nine, and that there was a year not above two hundred years ago wherein the cycle of the sun was eight, and the cycle of the moon ten: What was the number of the year?

N. B. If to any year of Christ be added 9, and the sum be divided by 28, the remainder, or 28 if nothing remains, is what is called the cycle of the sun for that year; and if 1 be added to any year of Christ, and the sum be divided by 19, the remainder, or 19 if nothing remains, is the cycle of the moon: hence if any year of Christ be severally divided by 28 and 19, and the remainders be d and e respectively, $d+9$, or $d+9-28$ will be the cycle of the sun, and $e+1$, or $e+1-19$ will be the cycle of the moon for that year: now in our case, $d+9$ cannot be equal to 8, for then d would be negative; make then $d+9-28=8$, and you will have $d=27$; moreover make $e+1=10$, and you will have $e=9$; therefore the question is now reduced to this; *What number is that, which being severally divided by twentyeight and nineteen, will have twentyseven and nine respectively remaining?*

Here $a=28$, $b=19$, $l=1$, $d=27$, $e=9$; and the quotients of a continual division from 28 and 19 are 1, 2 and 9, in number odd; strike
out

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out therefore the first and last, and with the remaining quotient 2 compute the last term of the series 0, 1, 2, and you will have $r=2$, and $\frac{ra}{1} \times d - e + d = 56 \times d - e + d$, or $57 \times d - e + e$, which is a general rule for finding the year from the two cycles given: restore now the values of d and e , that is, make $d=27$, and $e=9$, and you will have $56 \times d - e + d = 1035$, which number is too little for our purpose, and must be thus enlarged: the least common multiple of 28 and 19 is 532, in which revolution of years the cycles again return, and are both the same as they were 532 years before; add then one revolution of 532 years to 1035 the number above found, and the sum 1567 will be the true number of the year sought, as falling within the limits above prescribed: for $1567+9$, or 1576 being divided by 28, leaves 8 for the cycle of the sun; and $1567+1$, or 1568 being divided by 19, leaves 10 for the cycle of the moon.

PROBLEM 4.

207. To find as many numbers as we please with this property, to wit, that if any one of them be severally divided by three given divisors a , b and c , whereof a is supposed the greatest and c the least, the remainders shall be three given numbers d , e and f respectively.

SOLUTION.

Put A for the least common multiple of a and b ; then find (by the last problem) two numbers g and h of such a nature, that g being divided by a and b shall have d and e for remainders, and that h being divided by A and c shall have g and f for remainders; I say then that h will be one of the numbers that will answer the conditions: And when any one number of this kind is found, as many others as we please may be had by a continual addition and subtraction of the least common multiple of the three divisors a , b and c , found by art. 172.

The demonstration of this solution is easy; for since by the supposition a measures A , and A measures $h - g$, a measures $h - g$; but a measures also $g - d$; therefore a measures both $h - g$ and $g - d$, and consequently their sum $h - d$: in like manner, since b measures A , and A measures $h - g$, b measures $h - g$; but b measures also $g - e$; therefore b measures $h - e$: therefore since a measures $h - d$, and b measures $h - e$, it follows that if h be severally divided by a and b , the remainders will be d and e respectively: but if h be divided by c , the remainder is

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f by the supposition; therefore b is such a number as being divided by the three given divisors a , b and c , shall have the three given remainders d , e and f respectively. *Q. E. I.*

EXAMPLE I.

Let it be required to find a number, which being severally divided by 105, 40 and 36, shall have d , e and f for remainders respectively.

Now before we enter upon the solution of this problem, or any other of this nature, we are in the first place to enquire what restriction they are liable to, in order to judge of the possibility or impossibility of such problems: let such an enquiry be made here, and we shall find that if this problem be possible, the remainders d , e and f must be so restrained,

that $\frac{d-e}{5}$, $\frac{d-f}{3}$, and $\frac{e-f}{4}$ must all be whole numbers: this is evi-

dent from the solution of the foregoing problem; for 5 is the greatest common measure of a and b , 3 that of a and c , and 4 that of b and c : let us therefore proceed upon these suppositions, and see whether by their help we cannot form and demonstrate a general canon for the given divisors without calling in any further assistance. First then we are to find a number which being severally divided by the two first divisors 105 and 40, will have d and e for remainders; and such a one we shall find the number $64d - 63e$, as in the first example to the foregoing problem; call this g : then as the least common multiple of the two first divisors 105 and 40 is 840, we are again to find another number which being divided by 840 and the third divisor 36, will have g and f remaining; and such a one we shall find the number $141g - 140f$ to be, as in the second example to the same problem: make therefore $64d - 63e = g$, and $141g - 140f = b$, and b will be such a number as the problem requires.

If any one would have g struck out of this canon, and so have the value of b expressed by the help of the remainders d , e and f only, this is easily done: for since $g = 64d - 63e$, we shall have $141g - 140f = 9024d - 8883e - 140f = b$; divide this number $9024d - 8883e - 140f$ by 2520, (which being the least common multiple of all the three divisors, cannot affect the remainders in respect of these divisors,) and you will now have remaining a lesser value of b , equal to $1464d - 1323e - 140f$; to this value add $2520e + 2520f$ to make the parts all affirmative, and you will have $b = 1464d + 1197e + 2380f$, which is the least value of b that can be expressed in these general terms, till the remainders d , e and f come to be given in particular numbers.

The synthetical demonstrations of these canons have something very curious in them; and therefore I chose this example as full as I could, that

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that the general method of them might hereby be the better comprehended. We are then to demonstrate that the number b above found, to wit, $1464d + 1197e + 2380f$ is such a one as being severally divided by 105, 40 and 36 will have d , e and f remaining. Now to shew this, I first divide b by 105, and find the remainder to be $99d + 42e + 70f$; but the further to exhaust this remainder, in order to try whether the last remainder of this division will be d , I expunge the other two quantities $42e$ and $70f$ by the help of those two restrictions wherein the last remainder d is concerned, that is, by supposing both $\frac{d-e}{5}$ and $\frac{d-f}{3}$ to be whole numbers, expunging e by the former supposition wherein e is concerned, and f by the latter wherein f is concerned, thus: $42d - 42e$, or $42 \times \frac{d-e}{5}$, or $210 \times \frac{d-e}{5}$ is divisible by 105 without any remainder; therefore $42d - 42e$ may be added to $99d + 42e + 70f$ without affecting the subsequent remainders; add it then, and the sum will be $141d + 70f$: again, $70d - 70f$, or $210 \times \frac{d-f}{3}$ is divisible by 105, and therefore $70d - 70f$ may be added to $141d + 70f$, and the sum will be $211d$, which being divided by 105, there will remain d ; therefore b is such a number as being divided by 105 will have d for a remainder. In the next place I divide the number b , or $1464d + 1197e + 2380f$ by the next divisor 40, and find the remainder to be $24d + 37e + 20f$, where the last remainder is to be e ; therefore I exterminate the two quantities $24d$ and $20f$ by those two restrictions wherein the last remainder e is concerned, that is, wherein it is supposed that both $\frac{d-e}{5}$ and $\frac{e-f}{4}$ are whole numbers, exterminating d by the former supposition wherein d is concerned, and f by the latter wherein f is concerned, thus: $24d - 24e$, or $120 \times \frac{d-e}{5}$ is divisible by 40; therefore $24d - 24e$ may be subtracted from $24d + 37e + 20f$, and the remainder will be $61e + 20f$: again, $20e - 20f$, or $80 \times \frac{e-f}{4}$ is divisible by 40; therefore $20e - 20f$ may be added to $61e + 20f$, and the sum will be $81e$, which being divided by 40 will have e remaining; therefore b is such a number as being divided by 40 will have e for a remainder. Lastly I divide the number $1464d + 1197e + 2380f$ by 36, and the remainder is $24d + 9e + 4f$; but the last remainder is to be f ; therefore I exterminate $24d$ and $9e$ by the help of the two restrictions wherein f is concerned, that is, by supposing both $\frac{d-f}{3}$ and

and $\frac{e-f}{4}$ to be whole numbers, exterminating d by the former supposition wherein d is concerned, and e by the latter wherein e is concerned, thus: $24d-24f$, or $72 \times \frac{d-f}{3}$ is divisible by 36; therefore $24d-24f$ may be subtracted from $24d+9e+4f$, and there will remain $9e+28f$: again, $9e-9f$, or $36 \times \frac{e-f}{4}$ is divisible by 36; therefore $9e-9f$ may be subtracted from $9e+28f$, and there remains $37f$, which being divided by 36, there will remain f : therefore $1464d+1197e+2380f$ is such a number as being divided by 105, 40 and 36, will have for remainders d , e and f respectively. Q. E. D.

N. B. If the divisors had been made 60, 45 and 36, the example would have been as full, and in less numbers.

EXAMPLE 2.

Let the divisors be 9, 8 and 6: then because the divisors 9 and 8 are prime to each other, the restrictions will be reduced to these two, viz. that $\frac{d-f}{3}$ and $\frac{e-f}{2}$ must be both whole numbers; and the canon will be as follows: make $64d-63e=g$, and $13g-12f=b$, and b will be such a number as is required: strike g out of this canon, and you will have $832d-819e-12f=b$; divide by 72 (the least common multiple of the three given divisors) to reduce the expression, and then add $72e+72f$ to take off all negation, and you will have $40d+45e+60f$ for the least value of b that can be expressed in this form.

Now to try this theorem synthetically, I divide the number $40d+45e+60f$ by the first divisor 9, and the remainder is $4d+6f$, where d is to be the last remainder; and therefore $6f$ must be banished by the restriction which supposes $\frac{d-f}{3}$ to be a whole number, thus: $6d-6f$, or $18 \times \frac{d-f}{3}$ is divisible by 9; therefore $6d-6f$ may be added to $4d+6f$, and the sum will be $10d$, which being divided by 9 leaves d . Then I proceed to the next divisor 8, dividing $40d+45e+60f$ by 8, and the remainder is $5e+4f$, where e must be the last remainder; and therefore $4f$ must be banished upon the supposition that $\frac{e-f}{2}$ is a whole number, thus: $4e-4f$, or $8 \times \frac{e-f}{2}$ is divisible by 8; therefore $4e-4f$
may

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may be added to $5e + 4f$, and the sum will be $9e$, which being divided by 8 leaves e for a remainder. Lastly, I divide $40d + 45e + 60f$ by the last divisor 6, and the remainder is $4d + 3e$, both which quantities must be exterminated (because f is to be the last remainder) thus: $4d - 4f$.

or $12 \times \frac{d-f}{3}$ is divisible by 6; therefore $4d - 4f$ may be subtracted from $4d + 3e$, and there will remain $3e + 4f$; but $3e - 3f$, or $6 \times \frac{e-f}{2}$ is also divisible by 6; therefore $3e - 3f$ may be subtracted from $3e + 4f$, and there will remain $7f$, which being divided by 6 will have f for a remainder. Q. E. D.

EXAMPLE 3.

Let the divisors be 6, 5 and 4; and then the remainders will be liable to but one restriction, which is, that $\frac{d-f}{2}$ must be a whole number; and you will have the following canon, $25d - 24e = g$, and $16g - 15f = b$; whence exterminating g , you will have $400d - 384e - 15f = b$; divide by 60 (the least common multiple of the three given divisors) and to the remainder add $60e + 60f$ to make the parts of that denomination affirmative, and you will have $40d + 36e + 45f = b$. To try this, divide $40d + 36e + 45f$ by the first divisor 6, and the remainder will be $4d + 3f$; but $3d - 3f$, or $6 \times \frac{d-f}{2}$ is divisible by 6; therefore $3d - 3f$ may be added to $4d + 3f$, and the sum will be $7d$, which being divided by 6 leaves d for a remainder. Proceed now to divide $40d + 36e + 45f$ by the other two divisors 5 and 4 severally, and the remainders will come out e and f respectively, as will be found by a bare simple division only, without any other artifice.

From all these instances, and from the nature of the operations themselves we may see, that we shall never have a demand for more restrictions than those that are deduced at the beginning of the problem from the nature of the divisors; and that when no such restriction can be deduced, there will be no occasion for any, but the remainders will all come out by a simple division only, as will further appear in the next example.

EXAMPLE 4.

Let it be required to assign in what year of Christ the cycle of the sun was eight, the cycle of the moon ten, and the cycle of indiction ten.

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Here it must be known, that if to any year of Christ be added 3, and the sum be divided by 15, the remainder, or 15 if nothing remains, is called the cycle of indiction for that year: hence, and from the definitions of the other two cycles already given in the last example of the last article it appears, that if any year of Christ be divided by 28, 19 and 15 severally, and the remainders be d , e and f respectively, $d+9$, $e+1$ and $f+3$ will be the respective cycles of the sun, moon and indiction; but if these numbers happen to be greater than 28, 19 and 15, then $d+9-28$, $e+1-19$, and $f+3-15$ will be the respective cycles: therefore we have three equations for determining d , e and f , to wit, $d+9-28=8$, $e+1=10$, and $f+3=10$; whence $d=27$, $e=9$, and $f=7$; so that upon the whole matter, we are *To find a number, which being divided by 28, 19 and 15, will have 27, 9 and 7 for remainders*; but we shall still call the remainders d , e and f till we have computed a general canon for the numbers 28, 19 and 15, which canon can be under no restriction, nor subject to any exception, because the numbers 28 and 19 are prime to each other, and both to the third number 15. Here then $g=57d-56e$, and $h=1065g-1064f=60705d-59640e-1064f$; divide this number by 7980 (the least common multiple of the three given divisors) and the remainder will be $4845d-3780e-1064f$; add $7980e+7980f$ to take off all negation, and you will have $4845d+4200e+6916f$ the least of this form that will answer the conditions, and will equally serve to find any year of Christ for which the three cycles, and consequently d , e and f are given, or its place in the Julian period. As for example, instead of d , e and f substitute their values above found, to wit, 27, 9 and 7, and you will have $4845d+4200e+6916f=130815+37800+48412=217027$; which being divided by 7980 leaves 1567 for the year of Christ sought; to which if 9, 1 and 3 be severally added, and the sums divided by 28, 19 and 15, the remainders will be 8, 10 and 10 the three cycles proposed.

*The Julian period is a revolution of 7980 years, bearing date from before the beginning of the world, and is not yet passed over: Its character is, that if the number of any year of this period without any further preparation be divided by 28, 19 and 15, the remainders will be the three cycles of the sun, moon and indiction for that year: therefore to find what place the year enquired after hath in this period, is nothing else but to find a number, which being divided by 28, 19 and 15, will have the three cycles 8, 10 and 10 for remainders. Make then $d=8$, $e=10$ and $f=10$, and you will have $4845d+4200e+6916f=149920$; which being divided by 7980 leaves 6280; therefore the year sought is the 6280th year of the Julian period, which answers to the 1567th year of Christ. Whence we may observe by the bye, that *If to any year of Christ be added 4713,*
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the sum will be the correspondent year of the Julian period: the reason of this is, because the difference betwixt 1567 and 6280 is 4713.

SCHOLIUM.

From what has been laid down in the two last problems, and especially in this, I doubt not but the learner will be able to make his way through more complicated cases, when more divisors and more remainders are given, and consequently more restrictions necessary; nor do I doubt but that he will be able to apply these restrictions so effectually, as to adapt synthetical demonstrations to all cases.

These two last problems are such as men of numbers have a curiosity to look into; and therefore I hope to be excused if I have dwelt somewhat longer than ordinary upon them; having however omitted several observations relating thereto, which to some perhaps would not have been unacceptable, purely because I would not be too tedious to the generality of my readers.

PROBLEM 5.

208. *Of two given numbers a and b , whereof a is the greater, to find two multiples whose difference shall be any given number whatever that is divisible by the greatest common measure of a and b .*

SOLUTION.

Let d represent in general the difference of the multiples sought; let $a=13$, and $b=7$, and their least common multiple will be 91 by art. 171. This supposed, find by the third problem a number which being divided by a will have d remaining, and being divided by b will have nothing remaining; such a one will be the number $14d$, as appears from the fifth example of that problem; therefore $13d$ and $14d$ are two multiples of a and b respectively, which will answer the conditions of the problem. Again, as 91, and consequently $91d$ is a common multiple of both a and b , if $13d$, which is a multiple of a , be subtracted from $91d$, another multiple of a , the remainder $78d$ must also be a multiple of a by art. 169; and if $14d$, which is a multiple of b , be subtracted from $91d$, another multiple of b , the remainder $77d$ must also be a multiple of b ; therefore $78d$ and $77d$ are other two multiples of a and b respectively, which will solve the problem as well as the former; the former solve it when a 's multiple is intended to be less than that of b , the latter when a 's multiple is intended to be greater than that of b , and both when the matter is left indifferent. But if we would reduce the multiples in both cases to the least in their kind, that is, if in both cases we would find the least multiples of a and b whose difference is d , the

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leffer multiples in both cases, that is, $13d$ and $77d$ must be severally divided by the least common multiple of a and b , that is, by 91 : be it so, and let the remainders be m and n respectively: I say then, that m will be the least multiple of a which with d added can make a multiple of b , and that n will be the least multiple of b which with d added can make a multiple of a .

First, that the remainder m is a multiple of a is plain, because both the divisor 91 , and the dividend $13d$ were so. 2dly, That $m+d$ is a multiple of b is also evident; for if 91 being taken from $13d$ as often as possible leaves m , the same 91 taken the same number of times from $14d$ will leave $m+d$; and this $m+d$ must be a multiple of b , because both 91 and $14d$ were so. Lastly, that the multiples m and $m+d$ are the least in their kind, appears from the nature of division, and from art. 173; for the remainder m , if taken the least possible, can never be greater than the divisor, which was the least common multiple of a and b , nor even equal to it, unless to prevent there being no remainder at all; therefore the multiples m and $m+d$ are the least in their kind in the former case: and for the same reason the multiples n and $n+d$ will be the least in their kind in the latter. Thus we have found not only two sorts of multiples which will equally solve the problem, but also the least multiples of each sort. Q. E. I.

N. B. 1st. If m , or n , or both be equal to nothing, their places must be supplied by the least common multiple of a and b , as was hinted before. This can never happen but when either b , or a , or both measure d : if b measures d , and consequently $13d$, then $13d$ will be a multiple of b ; but $13d$ is also a multiple of a by the construction; therefore $13d$ will be a common multiple of both a and b , and as such, being divided by 91 their least common multiple, will have no remainder; therefore in this case the remainder m , if unsupplied, will be equal to nothing: in like manner, if a measures d , and consequently $77d$, then $77d$ will be a common multiple of a and b , and so n will be equal to nothing; therefore if both b and a measure d , both m and n will be equal to nothing.

2dly. If instead of the numbers 13 and 7 in the resolution of this problem we had made use of any other numbers in the same proportion, as 39 and 21 whose greatest common measure is 3 , the canon adapted to these numbers would have been the same with the former, that is, the multiples of 39 and 21 would have been expressed by $13d$ and $14d$ in the former case, and by $78d$ and $77d$ in the latter; but the synthetical demonstration would have been something different; for in this case

$$\frac{d}{3}$$

Art. 208, 209. PROBLEMS ADMITTING MANY ANSWERS. 331

$\frac{d}{3}$ must have been a whole number, and we must have argued thus:

$13d$, or $\frac{39d}{3}$ is a multiple of 39, and $14d$, or $\frac{42d}{3}$ is a multiple of 21, and therefore these multiples will answer the condition: again, $78d$ and $77d$, or $\frac{234d}{3}$ and $\frac{231d}{3}$ are multiples of 39 and 21 respectively, because the numbers 234 and 231 are so,

Of the solution here given take the following example. Let it be required to find two multiples of 13 and 7 whose difference shall be 500. Here $13d$ or 6500 being divided by 91 has 39 for a remainder; therefore $m=39$: again, $77d$ or 38500 divided by 91 has 7 for a remainder; therefore $n=7$; therefore 39 is the least multiple of 13 which with 500 will make 539 a multiple of 7; and on the other hand, 7 is the least multiple of 7 which with 500 will make 507 a multiple of 13.

PROBLEM 6.

209. Having given two numbers a and b whose least common multiple is c , and also a third number as d that is divisible by the greatest common measure of a and b ; It is required to find, if possible, two multiples of a and b whose sum shall be that given number d , or (which is the same thing) it is required to divide the given number d into two such parts, that one part shall be a multiple of a , and the other a multiple of b .

SOLUTION.

By the help of the last problem find m the least multiple of a which with d added can make a multiple of b ; and subtracting it from c the least common multiple of a and b , call the remainder r ; or if m before it's place be supplied, be equal to nothing, make $r=c$; then if r be greater than or equal to d , the problem will be incapable of any solution; but if r be less than d , I say then that r will be the least multiple of a , and $\overline{d-r}$ the greatest multiple of b that will solve the problem: and this first solution being obtained, if the problem admits of more, they will easily be had by a continual addition and subtraction of the least common multiple c , as $\overline{r+c}$ and $\overline{d-r-c}$, $\overline{r+2c}$ and $\overline{d-r-2c}$, $\overline{r+3c}$ and $\overline{d-r-3c}$, &c: therefore if $\overline{d-r}$ in the first solution be divided by c so as to have any remainder as s , the quotient which shews how often c can be subtracted from $\overline{d-r}$, will also shew how many solutions the problem will admit of after the first; and therefore this quotient in-

creased by unity will be the number of solutions in the whole; lastly I say that the number s will be the least multiple of b , and $d-s$ the greatest multiple of a that will solve the problem this way: so that the numbers r and $d-r$, as also $d-s$ and s may be called the extreme solutions; except when the problem admits of but one solution, and then these extremes will unite in that single solution.

DEMONSTRATION.

1st. Since by the construction both c and m are multiples of a , their difference $c-m$ or r will be a multiple of a by art. 169; and since both c and $m+d$ are multiples of b , their difference $m-c+d$ or $d-r$ will be a multiple of b .

2dly. When $d-r$ was divided by c , the remainder was s ; therefore c measures $d-r-s$: since then b measures c , and c measures $d-r-s$, b measures $d-r-s$; but b measures $d-r$ as was proved before; therefore b measures both $d-r$ and $d-r-s$, and consequently their difference s ; therefore s is a multiple of b : again, since a measures c , and c measures $d-r-s$, a measures $d-r-s$; but a measures r , as was proved before; therefore a measures both r and $d-r-s$, and consequently their sum $d-s$; therefore $d-s$ is a multiple of a .

3dly. We are in the last place to prove that r and $d-r$, as also $d-s$ and s are extreme solutions: and first, if any solution of this problem could be obtained wherein any less multiple of a than r was concerned, it must be had by subtracting the least common multiple c from r , and adding it to $d-r$, and then the multiples would be $r-c$ and $d-r+c$; but r or $c-m$ can never be greater than c , and therefore $r-c$ must either be nothing or negative, both which cases are excluded out of the problem; therefore r must be the least multiple of a , and $d-r$ the greatest multiple of b that will solve the problem: and by a like process it may be demonstrated that s is the least multiple of b , and $d-s$ the greatest multiple of a that will solve the problem.

An example to the foregoing solution.

Let it be required to divide the number 500 into two such parts that one part may be a multiple of 13, and the other of 7. Here $a=13$, $b=7$, $c=91$, and $d=500$, as in the last problem; and since m was there

Art. 209, 210. PROBLEMS ADMITTING MANY ANSWERS. 333

there found equal to 39, we shall have $\overline{c-m}$ or $r=52$, and $\overline{d-r}=448$; therefore 52 and 448 are the first numbers that will solve the problem, the former being a multiple of 13, and the latter of 7: divide 448 by 91, and the quotient will be 4, and the remainder 84; therefore 416 which is a multiple of 13, and 84 which is a multiple of 7 are the last numbers that will solve the problem. The solutions are five in number, and being placed in order will stand thus, the first number of every set being a multiple of 13, and the second of 7.

- 1st. 52 and 448,
- 2d. 143 and 357,
- 3d. 234 and 266,
- 4th. 325 and 175,
- 5th. 416 and 84.

LEMMA 12.

A THEOREM.

210. Let $\frac{a}{b}$ express any fraction in it's least terms, and let this fraction be multiplied by some whole number d , so that the product $\frac{ad}{b}$ may be also a whole number: I say then that the multiplicator d must either be equal to, or some multiple of the denominator b .

For since by the supposition, the fraction $\frac{ad}{b}$ is equivalent to some whole number, let that number be c ; then we shall have $c = \frac{ad}{b}$, and $\frac{c}{d} = \frac{a}{b}$; therefore the fraction $\frac{c}{d}$ must either be the same with the fraction $\frac{a}{b}$, or else it must be reducible to it, as to it's least terms, by the greatest common measure of c and d , since $\frac{a}{b}$ is supposed to be in it's least terms: if $\frac{c}{d}$ be the same with $\frac{a}{b}$, then d must be the same with b , which is one case of the lemma; if $\frac{c}{d}$ be equal to, but not the same with $\frac{a}{b}$, let e be the greatest common measure of c and d ; then will $\frac{c}{e} = a$, and $\frac{d}{e} = b$, and $d = eb$, in which case d is a multiple of b . Q. E. D.

COROLLARY

COROLLARY.

Hence I infer, that if $\frac{a}{b}$ expresses any fraction in it's least terms, the denominator b will be the least number that fraction can be multiplied by to reduce it to a whole number.

PROBLEM 7.

211. Let $a, b, c, d, e, \&c$ be a number of hands like those of a clock, all turning uniformly upon the same center, and all moving the same way; and let the same letters $a, b, c, d, e, \&c$ represent also their respective periodical times, or numbers proportionable to them: What will be the synodical period of the whole system, that is, supposing these hands to start all together from the same point as s , how long will it be before they all come together again for the first time, whether this conjunction happens in the point s , or in any other part of the circle wherein these motions are performed?

N. B. To say that of two hands a and b , a is got one circle and a half before b , or two circles and a half, or three circles and a half, is to say no more than that a is half a circle before b ; and to say that a is one circle or two circles or three circles before b , is the same thing as to say that the two hands a and b are coincident: And whenever any angular distance of this kind is expressed by a mixt number, or by an improper fraction, to reduce it (a phrase frequently made use of in the following solution) implies two things; first to throw away the integral part, and then to reduce the remaining fraction into it's least terms: thus the distance $3\frac{1}{2}$ after reduction becomes $\frac{1}{2}$; for it is the same thing to say that a is three circles and four sixth parts of a circle before b , as to say that a is two thirds of a circle before b , or one third behind it. This language being well understood and attended to, the solution will be as follows.

SOLUTION.

1st. Let p be the synodical period of the two first hands a and b : then to find the distance of the hand a from the point s at the end of the time p say, If in the time a the first hand makes one revolution, how many revolutions and parts of a revolution will it make in the time p ? and the answer is $\frac{p}{a}$; therefore the distance of the hand a from the point s at the end of the time p is $\frac{p}{a}$; and for the same reason the distance of the
hand

hand b from the point s is $\frac{p}{b}$; therefore the difference of these two frac-

tions, to wit, $\frac{p}{a} - \frac{p}{b}$ shews how much the hand a is advanced before the hand b during the time p ; but after the two hands a and b have departed from the point s , and parted from one another, they can never come together again till one is advanced an entire circle before the other; therefore if p be the synodical period of the two hands, we shall have $\frac{p}{a} - \frac{p}{b} = 1$, and $p = \frac{ab}{b-a}$: this is upon a supposition that the hand a is the swifter mover; but if b be the swifter mover, we shall then have $\frac{p}{b} - \frac{p}{a} = 1$, and $p = \frac{ab}{a-b}$; universally, p is the product of the two numbers a and b divided by their difference. And if p be the time at the end of which the two hands a and b will come first together, at the end of the time $2p$ they will come together a second time, and at the end of the time $3p$ a third time, and so on.

2dly. At the end of the time p , the distance of the hand a , and consequently of both the hands a and b from the point s will be $\frac{p}{a}$, as before; and at the same time the distance of the hand c from the same point s will be $\frac{p}{c}$; reduce therefore these two distances $\frac{p}{a}$ and $\frac{p}{c}$, and then taking their difference, reduce that also to it's least terms $\frac{n}{q}$; and then the distance of the hand c from the two coincident hands a and b at the end of the time p will be $\frac{n}{q}$, and at the end of the time $2p$, $\frac{2n}{q}$, and at the end of the time $3p$, $\frac{3n}{q}$, &c, and therefore at the end of the time pq the distance of the hand c from the coincident hands a and b will be n ; but n is a whole number, and implies a whole number of circles or revolutions; therefore at the end of the time pq the hand c will coincide with the other two a and b : and that this will be the first coincidence of the three hands a , b and c since their departure from s , is evident from the last article, the denominator q being the least number: the fraction $\frac{n}{q}$ can be multiplied by to make it a whole number; therefore the time pq is the synodical period of the three first hands a , b and c .

3dly.

3dly. At the end of the time pq , the distance of the hand a , and consequently of all the three hands a , b and c from the fixed point s is $\frac{pq}{a}$, and the distance of the hand d from the same is $\frac{pq}{d}$; reducing then the distances $\frac{pq}{a}$ and $\frac{pq}{d}$, let their difference after reduction be a fraction whose denominator is r when reduced to it's least terms, and the time pqr will be the synodical period of the four first hands a , b , c and d .

4thly. At the end of the time pqr the distance of a from s will be $\frac{pqr}{a}$, and the distance of e from s will be $\frac{pqr}{e}$, and if these distances be reduced, and their difference be also reduced to it's least denominator s , the time $pqrs$ will be the synodical period of the five first hands a , b , c , d , e : and so on. The solution then (to recapitulate) is as follows:

1st. Multiply the two periodical times a and b together, and then dividing their product by their difference, call the quotient p , whether it be a whole number or a fraction.

2dly. Reduce the two distances $\frac{p}{a}$ and $\frac{p}{c}$, and then taking their difference, reduce that also to it's least denominator q .

3dly. Reduce the distances $\frac{pq}{a}$ and $\frac{pq}{d}$, and their difference to it's least denominator r .

4thly. Reduce the distances $\frac{pqr}{a}$ and $\frac{pqr}{e}$, and their difference to it's least denominator s ; &c: I say then that p will be the synodical period of the two first hands, pq that of the three first, pqr that of the four first, $pqrs$ that of the five first, &c.

N. B. It matters nothing, as to the conclusion, what order the hands are taken in.

EXAMPLE.

Let $a=3$, $b=7$, $c=10$, $d=12$, $e=15$.

SOLUTION.

1st. $p = \frac{21}{4}$; therefore $\frac{p}{a}$ is $\frac{7}{4}$, and when reduced $\frac{3}{4}$; $\frac{p}{c}$ is $\frac{21}{40}$, and the difference between $\frac{3}{4}$ and $\frac{21}{40}$ is $\frac{9}{40}$; therefore $q = 40$.

2dly.

Art. 211, 212. PROBLEMS ADMITTING MANY ANSWERS. 33

2dly. Therefore $pq = \frac{21}{4} \times 40 = 210$; therefore $\frac{pq}{a}$ is 70, and when reduced 0; $\frac{pq}{d}$ is $\frac{210}{12}$, and when reduced $\frac{1}{2}$, and the difference betwixt 0 and $\frac{1}{2}$ is $\frac{1}{2}$; therefore r is 2.

3dly. Therefore $pqr = 420$; therefore $\frac{pqr}{a}$ is 140, and when reduced 0; $\frac{pqr}{e}$ is 28, and when reduced 0, and the difference between 0 and 0 is 0, or $\frac{0}{1}$; therefore s is 1. So that the synodical period of the two first hands is $\frac{21}{4}$, that of the three first 210, that of the four first 420, and that of the five first 420; which shews that when d joined the three first hands a , b and c , e joined them at the same time.

If the hand b should move a contrary way to that of a , then the equation which determines p will not be $\frac{p}{a} - \frac{p}{b} = 1$ as before, but $\frac{p}{a} + \frac{p}{b} = 1$; whence p will be found equal to $\frac{ab}{a+b}$; for the hands a b must now be looked upon as lying on contrary sides of the point s , in which case the sum of their distances from this point will be their distance from each other: again, if c moves contrary to a , the distances $\frac{p}{a}$ and $\frac{p}{c}$ must be reduced as before; but then it must be their sum, and not their difference, which being reduced to it's least terms will give the denominator q ; and so on.

PROBLEM 8.

212. Let there be three unknown quantities x , y and z , whose relation to each other is expressed by the two following equations,

$$x + 2y + 3z = 20, \text{ and}$$

$$4x + 5y + 6z = 47:$$

It is required to find the values of x , y and z in integral and affirmative numbers.

SOLUTION.

Equation 1st, $x + 2y + 3z = 20.$

2d, $4x + 5y + 6z = 47.$

Subtract the second equation from four times the first, and you will have $3y + 6z = 33$; whence

Equation 3d, $y = 11 - 2z.$

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Subtract

Subtract five times the first equation from twice the second, and you will have $3x - 3z = -6$, and

Equation 4th, $x = z - 2$.

And thus we have the values of x and y with respect to z , and z itself is left undetermined because there were but two equations to three unknown quantities: and that these are the true values of x and y will sufficiently appear by substituting $z - 2$ and $11 - 2z$ instead of x and y in the first and second equations; for upon that supposition we shall have in the first equation $x + 2y + 3z = z - 2 + 22 - 4z + 3z = 20$, and in the second equation we shall have $4x + 5y + 6z = 4z - 8 + 55 - 10z + 6z = 47$. Now as to the values of x , y and z , it appears from the third and fourth equations that whatever whole number is substituted for z , x and y must necessarily come out whole numbers; for if z be a whole number, not only $z - 2$ or x will be a whole number, but also $11 - 2z$ or y ; therefore if the problem was subject to no other limitations than barely that x , y and z should be whole numbers, the number of solutions would be infinite: but besides their being whole numbers, it is particularly specified in the problem that they shall be affirmative numbers; and therefore to find out the number of solutions in this case, we must again attend to the values of x and y in the two last equations: now since in the fourth equation $x = z - 2$, it is plain that if x be affirmative, z must be taken greater than 2: again, since in the third equation $y = 11 - 2z$, if y be affirmative, $2z$ must be less than 11, and z less than $5\frac{1}{2}$; whence it follows *a fortiori* that z must be less than 6; therefore that x and y as well as z may be affirmative, the value of z must consist between these two limits, *viz.* 2 and 6; therefore if all the three quantities x , y and z be affirmative, there are but three numbers that can be substituted for z , to wit, the numbers 3, 4 and 5; so that by the last limitation of the problem, all the infinite number of solutions it would otherwise have admitted of, are now reduced to three, and are thus determined: Since any of the three numbers 3, 4 and 5 may be put for z , let $z = 3$; then we shall have $z - 2$ or $x = 1$, and $11 - 2z$ or $y = 5$; so that the numbers x , y and z will be 1, 5 and 3 respectively, and will answer the conditions of the first and second equations; for upon this supposition $x + 2y + 3z = 1 + 10 + 9 = 20$, and $4x + 5y + 6z = 4 + 25 + 18 = 47$. Let $z = 4$; then we have $z - 2$ or $x = 2$, $11 - 2z$ or $y = 3$, and the numbers will be 2, 3 and 4, which will also answer; for in this case $x + 2y + 3z = 2 + 6 + 12 = 20$, and $4x + 5y + 6z = 8 + 15 + 24 = 47$. Let $z = 5$; then will $z - 2$ or $x = 3$, $11 - 2z$ or $y = 1$, and the numbers will be 3, 1 and 5; here then again we have $x + 2y + 3z = 3 + 2 + 15 = 20$, and $4x + 5y + 6z = 12 + 5 + 30 = 47$.

x	y	z .
1	5	3.
2	3	4.
3	1	5.

LEMMA 13.

A PROBLEM.

213. Let a , b and c be three fixed or determinate whole numbers, where-
of a and b are prime to each other; and let x and y be any two vari-
able or indeterminate whole numbers, whose relation to each other is
constantly expressed by the equation $ax \pm by = c$: What will be the
next whole number values of x and y , whose relation to each other can
be expressed by the same equation?

CASE 1.

Let the equation be $ax - by = c$, and let the new values of x and y
be $x + d$ and $y + e$; then since the relation of these two values is to be
expressed by the same equation as was that of the former, the equation
will be $ax + ad - by - be = c$; but $ax - by = c$ *ex hypothesi*; there-
fore $ad - be = 0$, and $be = ad$, and $\frac{e}{d} = \frac{a}{b}$; therefore if e and d be
any two numbers taken in the same proportion as a to b , the values $x + d$
and $y + e$ will fall under the same relation with x and y : but to find the
nearest values to x and y , e and d must not only be in the same propor-
tion with a and b , but they must be the least in their proportion, that
is, the fraction $\frac{e}{d}$ must be in it's least terms: since then the two frac-
tions $\frac{a}{b}$ and $\frac{e}{d}$ are equal in value, and both in their least terms, they
must also be equal in terms, that is, e must be equal to a , and d to b ;
therefore the nearest values to x and y in the same relation are $x + b$
and $y + a$, or $x - b$ and $y - a$.

CASE 2.

Let the equation be $ax + by = c$, and let $x + d$ and $y + e$ be the
nearest values of x and y in the same relation, and the equation will be
 $ax + ad + by + be = c$; but $ax + by = c$ *ex hypothesi*; therefore ad
 $+ be = 0$; therefore $be = -ad$, and $\frac{e}{d} = -\frac{a}{b}$; therefore if e and d

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be

be the least in their proportion, as they ought to be, either e must be equal to $-a$ and d to b , or e to $+a$ and d to $-b$; therefore the two nearest values to x and y in the same relation will be $x+b$ and $y-a$, or $x-b$ and $y+a$.

COROLLARY.

Hence, if the values of x be taken in an arithmetical progression whose common difference is b the coefficient of y , the respective values of y will also form an arithmetical progression whose common difference is a the coefficient of x : and if the equation be $ax-by$ &c, both progressions will increase, or both decrease together; but if the equation be $ax+by$ &c, and if the values of x form an increasing progression, those of y will form a decreasing one, and vice versa.

PROBLEM 9.

214. It is required to divide a hundred into three such parts x , y and z , that $9x+15y+20z$ may make fifteen hundred.

SOLUTION.

$$\text{Equation 1st, } x+y+z=100.$$

$$2d, 9x+15y+20z=1500.$$

Subtract nine times the first equation from the second, and you will have $6y+11z=600$; whence

$$\text{Equation 3d, } y=100-\frac{11z}{6}.$$

Subtract the second equation from fifteen times the first, and you will have $6x-5z=0$; whence

$$\text{Equation 4th, } x=\frac{5z}{6}.$$

And if these values of x and y in the two last equations be substituted instead of x and y in the two first, they will answer the conditions of those equations, as will easily appear upon trial.

Now as to the determination of the values of x and y , where $x=\frac{5z}{6}$

and $y=100-\frac{11z}{6}$, it is evident that if for z be substituted any number that is divisible by 6 without a remainder, that is, any term of this arithmetical progression 6, 12, 18, 24, &c *ad infinitum*, the quantity x will come out a whole number and affirmative; but as $y=100-\frac{11z}{6}$, it is evident that y cannot be affirmative unless $\frac{11z}{6}$ be less than

100,

Art. 214, 215. PROBLEMS ADMITTING MANY ANSWERS. 341

100, that is, unless $11z$ be less than 600, or z less than $54\frac{6}{11}$; therefore z must not be made equal to any whole number that is greater than 54; therefore z may be made equal to any multiple of 6 from 6 to 54 inclusively. Now as the equation expressing the relation betwixt x and z was $6x - 5z = 0$, it follows from the last article, that if the values of z be taken in an increasing arithmetical progression whose common difference is 6, the correspondent values of x will form an increasing arithmetical progression whose common difference is 5: again, since the equation expressing the relation betwixt y and z was $6y + 11z = 600$, it follows that if the values of z be made to increase in an arithmetical progression whose common difference is 6, those of y will sink in an arithmetical progression whose common difference is 11. Make $z = 6$, and $100 - \frac{11z}{6}$ or y will be 89, and $\frac{5z}{6}$ or x will be 5, and the three parts sought will be 5, 89 and 6, which will answer the conditions.

As to the other solutions, (for they are nine in all,) these are easily found by the observations above, and are put down in the following table:

x	y	z
5	89	6
10	78	12
15	67	18
20	56	24
25	45	30
30	34	36
35	23	42
40	12	48
45	1	54

PROBLEM 10.

215. Let it be required to divide the number twentyfour into three such parts x , y and z , that $x + 8y + 12z$ may make two hundred and one.

SOLUTION.

Equation 1st, $x + y + z = 24$.

2d, $x + 8y + 12z = 201$.

Subtract the first equation from the second, and you will have $7y + 11z = 177$; whence

Equation 3d, $y = \frac{177 - 11z}{7}$.

Subtract

Subtract the second equation from eight times the first, and you will have $7x - 4z = -9$; whence

$$\text{Equation 4th, } x = \frac{4z - 9}{7}.$$

From the third and fourth equations it appears, that to make x and y whole numbers as well as z , some such number must be substituted for z as will make not only $\frac{177 - 11z}{7}$, but also $\frac{4z - 9}{7}$ a whole number:

now had both $\frac{177}{7}$ and $\frac{9}{7}$ been whole numbers, this problem might have been solved as was the last; but as it happens otherwise, we must proceed after another manner; but first we ought to enquire whether these two conditions, *viz.* that both $\frac{177 - 11z}{7}$ and $\frac{4z - 9}{7}$ are whole numbers, are consistent with each other or not; for if they be found inconsistent, then it will be impossible for both x and y together to be whole numbers, and the problem will be impossible: divide then $4z - 9$ by 7, which in the present case is no more than subtracting 7 from $4z - 9$, and the remainder will be $4z - 2$; divide also $177 - 11z$ by 7, and the remainder will be $2 - 4z$; and therefore the enquiry is now reduced to this, whether it be possible for $\frac{4z - 2}{7}$ and $\frac{2 - 4z}{7}$ to be both whole numbers; and this supposition will be so far from being impossible, that one part necessarily implies the other, these two numbers being the same affirmatively and negatively taken, and therefore they must either be both or neither of them whole numbers. This matter being settled, we are in the next place to enquire, what will be the least number which being put for z will make $\frac{4z - 2}{7}$ a whole number: now in order to this, make 7 the denominator equal to a , and 4 the coefficient of z in the numerator equal to b , and from these values of a and b derive a series of equations as in the 175th article, and they will be as follows:

$$1a - 0b = +7.$$

$$0a - 1b = -4.$$

$$a - b = +3.$$

$$a - 2b = -1.$$

$$3a - 5b = +1.$$

Of the two last equations take that whose absolute term is negative, to wit, $a - 2b = -1$, because -2 the numeral part of the numerator

$$4z - 2$$

Art. 215, 216. PROBLEMS ADMITTING MANY ANSWERS. 343

$4z-2$ is negative; and then multiplying the equation $a-2b=-1$ by 2 that numeral part taken affirmatively, you will have $2a-4b=-2$ the simplest equation of it's kind: make 4 the coefficient of $-b$ in this last equation equal to z , and restoring now the values of a and b , to wit, 7 and 4, the equation will be changed into this, $14-4z=-2$, and you will have $\frac{4z-2}{7}=2$, a whole number; therefore if the whole number 4 be put for z , the other two parts x and y will also be whole numbers, and you will have x or $\frac{4z-9}{7}=1$, and y or $\frac{177-11z}{7}=19$, and therefore the parts x, y and z will be 1, 19 and 4 respectively. Now to know in what progressions the values of x, y and z will vary, since the equation between x and z was $7x-4z=-9$, it follows from art. 213, that if the values of z be continually increased by 7, the values of x will be continually increased by 4; and since the equation between y and z was $7y+11z=177$, it follows also that if the values of z be continually increased by 7, those of y will be continually diminished by 11; therefore since the first value of x is 1; it's several values will be 1, 5, 9, 13, &c; the several values of y will be 19, 8, -3 , -14 , &c; and lastly those of z will be 4, 11, 18, 25, &c: whence it appears that this problem is not capable of above two solutions, because if the solutions be continued any further, y will be negative: here then we have $x=1, y=19, z=4$; or $x=5, y=8$, and $z=11$.

PROBLEM II.

216. Suppose one would buy forty birds, consisting of partridges, larks and quails for ninetyeight pence, paying three pence apiece for the partridges, halfpence apiece for the larks, and four pence apiece for the quails: The question is, how many he must have of each sort.

SOLUTION.

Put x for the number of partridges, y for the number of larks, and z for the number of quails; then will $3x, \frac{y}{2}$, and $4z$ express the number of pence to be given for each sort; and the problem abstracted from words will stand thus: If $x+y+z=40$, and $3x+\frac{y}{2}+4z=98$, or $6x+y+8z=196$; what are x, y and z ?

Equation 1st, $x+y+z=40$.

2d, $6x+y+8z=196$.

Subtract

Subtract the second equation from six times the first, and you will have $5y - 2z = 44$; whence

$$\text{Equation 3d, } y = \frac{2z + 44}{5}.$$

Subtract the first equation from the second, and you will have $5x + 7z = 156$; whence

$$\text{Equation 4th, } x = \frac{156 - 7z}{5}; \text{ therefore } \frac{2z + 44}{5} \text{ and } \frac{156 - 7z}{5}$$

must both be whole numbers, the possibility whereof I thus demonstrate: divide $156 - 7z$ by 5, and the remainder will be $1 - 2z$; divide $2z + 44$ by 5, and the remainder will be $2z + 4$, from which again subtracting 5, the remainder will be $2z - 1$; compare then these two remainders $2z - 1$ and $1 - 2z$, and the question will now be reduced to

this, *viz.* whether if $\frac{2z - 1}{5}$ be a whole number, $\frac{1 - 2z}{5}$ can be so too;

and the answer is ready, that if the former be a whole number, the latter must be so, since it is but the negative of the former: therefore if

$\frac{2z + 44}{5}$ be a whole number, $\frac{156 - 7z}{5}$ must be so too. This being

determined, let us now resume one of the former remainders, to wit, $2z - 1$, and let us enquire into the least number which being substituted for z will make $\frac{2z - 1}{5}$ a whole number: and this might be found

out after the same manner as in the last problem; but because the denominator 5 is but a small number, I rather chuse to do it by tryal (since four tryals at most must determine the question,) that is, I make z equal

to 1, 2, 3 and 4 successively, and try in which of these cases $\frac{2z - 1}{5}$

will be a whole number, and I find it will succeed when z is made equal

to 3; for then $\frac{2z - 1}{5} = 1$.

Note. The number of tryals must always be less by unity than the denominator.

Thus then we have found $z = 3$; whence y or $\frac{2z + 44}{5}$ will be 10,

and x or $\frac{156 - 7z}{5}$ will be 27; therefore if he buys twentyseven par-

tridges, ten larks, and three quails, he will have forty birds for ninety-

eight pence, as the problem requires.

As

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As to the other solutions, since the equation for x and z was $5x + 7z = 156$, and the equation for y and z was $5y - 2z = 44$, it follows from art. 213, that the successive values of z must be increased by a continual addition of 5, those of y by a continual addition of 2, and those of x must be diminished by a continual subtraction of 7; whence we have the following solutions:

x	y	z
27	10	3.
20	12	8.
13	14	13.
6	16	18.

Besides the four solutions of this problem exhibited in the table it will be impossible to assign any others; for should this table be carried on but one degree lower, the value of x would be found to be -1 ; and should it be taken but one degree higher, the value of z would be found to be -2 .

In my solution of this problem I left the value of the last quantity z undetermined, and reduced those of x and y to that, as usual: but if I had left the middle quantity y undetermined, and had reduced the values of x and z to the quantity y , the solution in this particular case would have been somewhat easier, and might have been obtained in the same manner as in the last problem but one, thus: subtract the second equation from six times the first, as they stand in the foregoing solution, and you will have $5y - 2z = 44$; whence $z = \frac{5y}{2} - 22$; therefore if z be

affirmative, $\frac{5y}{2}$ must be greater than 22, and $5y$ greater than 44, and y greater than $8\frac{4}{5}$; therefore if z be affirmative, y must be greater than 8: again, subtract the second equation from eight times the first, and you will have $2x + 7y = 124$, and $x = 62 - \frac{7y}{2}$; therefore $\frac{7y}{2}$ must be less than 62, and $7y$ less than 124, and y less than $17\frac{4}{7}$; therefore if x be affirmative, y must be less than 18; therefore all the values of y must lie between 8 and 18: but $\frac{7y}{2}$ in the expression of x , and $\frac{5y}{2}$ in the expression of y

shew that $\frac{y}{2}$ must be a whole number, or (which comes to the same thing) that y must be an even number; therefore all the even numbers between 8 and 18 may be put for y , to wit, 10, 12, 14, 16: let $y = 10$, and we shall have x or $62 - \frac{7y}{2} = 27$, and z or $\frac{5y}{2} - 22 = 3$; and hence may all the other values of x and z be found as in the foregoing table.

X x

Pro-

PROBLEM 12.

217. Suppose one would buy twenty birds for twenty pence, to wit, ducks at twopence apiece, partridges at halfpence apiece, and geese at threepence apiece: How many must he have of each sort?

SOLUTION.

Put x, y and z for the number of ducks, partridges and geese respectively, and consequently $2x, \frac{y}{2}$ and $3z$ for the price of each sort in pence, and you will have these two fundamental equations; $x+y+z=20$, and $2x+\frac{y}{2}+3z=20$, or $4x+y+6z=40$; whence

$$\text{Equation 1st, } x+y+z=20.$$

$$2\text{d, } 4x+y+6z=40.$$

Expunge z by subtracting the second equation from six times the first, and you will have $2x+5y=80$, and

$$\text{Equation 3d, } x=40-\frac{5y}{2}.$$

Therefore $\frac{5y}{2}$ must be less than 40, and $5y$ less than 80, and y less than 16. Again, expunge x by subtracting the second equation from four times the first, and you will have $3y-2z=40$, and

$$\text{Equation 4th, } z=\frac{3y}{2}-20; \text{ whence } \frac{3y}{2} \text{ must be greater than } 20,$$

and $3y$ greater than 40, and y must be greater than 13, and less than 16:

but $\frac{5y}{2}$ in the third equation, and $\frac{3y}{2}$ in the fourth shew that y must be an even number; and there is but one even number between 13 and 16, to wit 14; therefore y must be 14, and the problem will admit but of

one solution; therefore x or $40-\frac{5y}{2}$ must be 5, and z or $\frac{3y}{2}-20$ must be 1; that is, there must be 5 ducks, 14 partridges, and 1 goose.

Monfieur Bachet in his comment upon the forty first proposition of the fourth book of *Diophantus*, cites an epigram containing this problem, which (such as it is) I have here transcribed.

Ut tot emantur aves, bis denis utere nummis;

Perdix, anser, anas empta vocetur avis:

Sit simplex obolus pretium perdicis, ematur

Sex obolis anser, bisque duobus anas.

Art. 217, 218, 219. PROBLEMS ADMITTING MANY ANSWERS. 347

*Ut tua procedat in lucem quæstio, mentem
 Consule; sic loquitur pectoris arca mihi:
 Sint anates tres atque duæ, simplex erit anser,
 Accipe perdices quatuor atque decem.*

PROBLEM 13.

218. *Twenty persons consisting of men, women and children at a collation paid twenty shillings, the men paying four shillings apiece, the women sixpence apiece, and the children threepence apiece: How many were there of each sort?*

SOLUTION.

Put x , y and z for the number of men, women and children respectively, and consequently $4x + \frac{y}{2} + \frac{z}{4}$ for the number of shillings paid by them; and this last condition furnishes us with the following equation, $4x + \frac{y}{2} + \frac{z}{4} = 20$: multiply the whole equation by 4, to take off the fractions, and you will have $16x + 2y + z = 80$; and the equations will stand thus:

$$\text{Equ. 1st, } x + y + z = 20.$$

$$2d, 16x + 2y + z = 80.$$

Expunge z by subtracting the first equation from the second, and you will have $15x + y = 60$; whence

$$\text{Equ. 3d, } y = 60 - 15x.$$

Therefore $15x$ must be less than 60, and x must be less than 4. Exterminate y by subtracting twice the first equation from the second, and you will have $14x - z = 40$; whence

$$\text{Equ. 4th, } z = 14x - 40.$$

Therefore $14x$ is greater than 40, and x is greater than 2; therefore x must lie between 2 and 4. Now as the values of y and z were expressed in the third and fourth equations without fractions, it follows, that whatever whole number is substituted for x , y and z will come out whole numbers; but they will not come out affirmative, unless x be a whole number between 2 and 4; therefore x must be 3, and the problem admits of but one solution; therefore y or $60 - 15x$ will be 15, and z or $14x - 40$ will be 2: so there were three men, fifteen women, and two children.

PROBLEM 14.

219. *Forty one persons consisting of men, women and children at a collation paid forty shillings, the men paying four shillings apiece, the women three shillings apiece, and the children fourpence apiece: How many were there of each sort?*

X x 2

SOLU-

SOLUTION.

Put x , y and z for the number of men, women and children respectively, and consequently $4x + 3y + \frac{z}{3}$ for the number of shillings paid by them, and you will have $4x + 3y + \frac{z}{3} = 40$, or $12x + 9y + z = 120$; therefore

Equation 1st, $x + y + z = 41$.

2d, $12x + 9y + z = 120$.

Subtract the second equation from twelve times the first, and you will have $3y + 11z = 372$, and

Equation 3d, $y = 124 - \frac{11z}{3}$.

Therefore $\frac{11z}{3}$ must be less than 124, and $11z$ less than 372, and z must be less than 34. Subtract nine times the first equation from the second, and you will have $3x - 8z = -249$; whence

Equation 4th, $x = \frac{8z}{3} - 83$.

Therefore $\frac{8z}{3}$ must be greater than 83, and $8z$ greater than 249, and z must be greater than 31. Hence, and from the third and fourth equations, if x and y be whole numbers and affirmative, z must be some multiple of 3 lying between 31 and 34; but there is only one such multiple, to wit 33; therefore the problem admits but of one solution, and $z = 33$; therefore x or $\frac{8z}{3} - 83 = 5$, and y or $124 - \frac{11z}{3} = 3$: so there were 5 men, 3 women, and 33 children.

PROBLEM. 15.

220. It is required to divide the number thirty into three such integral parts x , y and z , that $2x + 9y + 15z$ may make four hundred and nineteen.

SOLUTION.

Equation 1st, $x + y + z = 30$.

2d, $2x + 9y + 15z = 419$.

Subtract twice the first equation from the second, and you will have $7y + 13z = 359$; whence

Equation 3d, $y = \frac{359 - 13z}{7}$.

Therefore z must be less than 28, or (which is the same thing) z must not be

Art. 220. PROBLEMS ADMITTING MANY ANSWERS. 349

be any whole number greater than 27. Subtract the second equation from nine times the first, and you will have $7x - 6z = -149$; therefore

$$\text{Equation 4th, } x = \frac{6z - 149}{7}.$$

Therefore z must not be any whole number less than 25. From the third and fourth equations it appears, that $\frac{359 - 13z}{7}$ and $\frac{6z - 149}{7}$ must both be whole numbers, the possibility whereof might be demonstrated as in the tenth and eleventh problems; but for the sake of variety I shall make it appear thus: $\frac{359 - 13z}{7}$ and $\frac{6z - 149}{7}$ when added together make $\frac{210 - 7z}{7}$, which is reducible to a whole number; therefore if $\frac{6z - 149}{7}$ be a whole number, the other $\frac{359 - 13z}{7}$ must be so too; for otherwise their sum could not be a whole number. To find then the value of z , I argue thus: whatever value of z will make $\frac{6z - 149}{7}$ a whole number, the same will make $\frac{6z - 2}{7}$ a whole number, because $6z - 149$ being divided by 7, there remains $6z - 2$; but 5 is a number which being substituted for z will make $\frac{6z - 2}{7}$ a whole number; therefore if z be made equal to 5, x and y will both come out whole numbers, but not both affirmative; for that both may be affirmative, it is required that z should not be any whole number less than 25, or greater than 27: however having found the least value of z that will make x and y whole numbers, to wit, 5; and finding by either of the equations $7x - 6z = -149$, or $7y + 13z = 359$ that all the other values of z must be found by a continual addition of the number 7, I begin a progression from 5, and carry it on by a continual addition of the number 7 till I come at a term that lies between the two limits of z (which will always be possible when the problem is possible) thus: 5, 12, 19, 26: since then the term 26 lies between the two limits of z , to wit, 24 and 28, I conclude that if I make z equal to 26, x and y will both come out whole numbers and affirmative; therefore $z = 26$, x or $\frac{6z - 149}{7}$ $= 1$, and y or $\frac{359 - 13z}{7} = 3$: so the numbers are 1, 3 and 26; which

upon

upon tryal will answer the conditions of the problem, and the problem admits only of this solution.

x	y	z .
—17	42	5.
—11	29	12.
—5	16	19.
1	3	26.
7	—10	33.

PROBLEM 16;

Being a general problem.

221. To find, if possible, three numbers, all integral and affirmative, whose sum is not only given, but also the sum of their products when severally multiplied by three given multipliers.

SOLUTION.

Let x , y and z be the three numbers sought, putting x for that number whose multiplier is the greatest, and z for that whose multiplier is the least, or negative if any of them be negative, or if two of them happen to be negative, put z for that number whose multiplier is the greater negative; then putting y for the middle term, you will have two equations, one whose absolute term is the given sum of the three numbers sought, and another whose absolute term is the given sum of the products. By the help of these two equations expunge z thus: multiply the first equation by the coefficient of z in the second, and then subtracting that product from the second equation, you will fall into a third of this form, $Ax + By = C$: divide this whole equation by the greatest common measure of A and B , and let the result be $ax + by = c$; then from this equation, and from the manner of deriving it, I shall deduce the following observations.

1st. If the directions given for the notation be duly observed, the numbers a and b will always be integral, affirmative, and prime to each other, and a will always be greater than b .

2d. If the numbers x and y be integral and affirmative, the number $ax + by$ or c must be so too; though it does not follow *e converso*, that if c be integral and affirmative, x and y must both be affirmative.

3d. If any one value of y be known, the correspondent values of x and z will easily be had by making $x = \frac{c - by}{a}$ and $z = s - x - y$, putting

s for

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s for the given sum of the three numbers sought: whence it follows, that if x and y be whole numbers, z must necessarily be so too.

4th. If the least value of y be known, and the problem admits of more solutions than one, as many other successive values of y as we shall have occasion for may easily be obtained by a continual addition of the number a ; whence the correspondent values of x will form a decreasing arithmetical progression whose common difference is b , and those of z another decreasing arithmetical progression whose common difference is $a-b$. The two first parts of my assertion are evident from art. 213, and the last I thus demonstrate: If the values of y increase by a and those of x decrease by b , the several values of the sum $x+y$ considered as one number will increase by $a-b$; but as the sum $x+y$ increases, the third quantity z must be diminished, because in the same problem the whole sum $x+y+z$ is always the same; therefore the successive values of z will form a decreasing arithmetical progression whose common difference is $a-b$.

5th. From this last observation it follows, that the raising of y is the sinking of x and z , and *vice versa*; inasmuch that when y is the least in it's kind, x and z will be the greatest in their's to solve the problem; and if in this state both be affirmative, they will do it; but if either or both happen to be negative, the problem will be impossible, and can have no solution.

Being thus prepared, I now proceed to find two such whole numbers as, being substituted for x and y , will make $ax+by=c$; which may be effected in the manner following: by the quotients of a continual division from a and b , form a set of equations according to art. 175; and as the original numbers a and b are prime to each other, the absolute term of the two last equations will be ± 1 ; take that equation wherein -1 is concerned, that is, take the last equation that stands in an even place, and it will be of this form, $ae-bd=-1$; change the signs, and you will have $-ae+bd=+1$; multiply all by c , and you will have $-ace+bcd=c$: make $-ce=x$ and $+cd=y$, and you will have two whole numbers x and y with this property, that $ax+by$ will be equal to c : but here as x is negative, it will be proper to try to raise it (if possible) to an affirmative number by sinking the value of y as low as we can, that is, by throwing away the number a from cd as often as we can, that is, by dividing cd by a ; for if this be done, the remainder if there be any, or the divisor a if there be none, will be the least value of y ; and if by this means x and z come out affirmative, the problem will admit of one or more solutions in affirmative numbers; but if x still happen to be negative, or if z comes out negative, there is

no

no raising these negative values without sinking the value of y , which is already as low as it can be, and therefore in this case the problem can have no affirmative solution.

As for the number d , since that is the coefficient of b in the last equation that stands in an even place, and since these coefficients are nothing else but the terms of a series begun from 0 and 1, and continued by the quotients of a continual division made from a and b , it is evident that if such a division be actually made, and the number of quotients be even, an unit must be deducted from the last, for a reason given in art. 175; but if the number of quotients be odd, the last must be thrown away, as leading to a wrong equation; and then if a series from 0 and 1 be computed by the rest of the quotients, the last term of the series will be d ; therefore in practice the number d may be computed by the help of this series only, without meddling with the other parts of the equations wherein these terms are concerned.

EXAMPLE I.

Let the equations be

$$1^{\text{st}}, \quad x + y + z = s.$$

$$2^{\text{d}}, \quad 10x + 5y + 2z = r.$$

Then subtracting twice the first equation from the second, you will have $8x + 3y = r - 2s$; and because the numbers 8 and 3 are prime to each other, the equation can be reduced no lower; therefore $a = 8$, $b = 3$, $c = r - 2s$, and $d = 3$; for the quotients of a continual division from 8 and 3 will be 2, 1, 2; whence, dropping the last because their number is odd, with the remaining quotients 2 and 1 form the series 0, 1, 2, 3, and you will have $d = 3$, and $cd = 3c$; thus you will have a particular canon adapted to the multipliers 10, 5 and 2, as follows: make $r - 2s = c$; divide $3c$ by 8, and the remainder will be y ; whence $x = \frac{c - 3y}{8}$, and $z = s - x - y$. As for instance, let $s = 20$ and $r = 53$,

that is, let it be required to divide the number 20 into three such parts, that ten times the first part, five times the second, and twice the third may all together make 53: here $r - 2s$ or $c = 13$, and $3c = 39$, which being divided by 8 leaves 7 for a remainder; therefore in this case $y = 7$, $\frac{13 - 3y}{8}$ or $x = -1$, and $s - x - y$ or $z = 14$: so the numbers are

-1 , $+7$ and $+14$; therefore this problem cannot be resolved in affirmative numbers, but in others it may; for $-1 \times 10 + 7 \times 5 + 14 \times 2$, that is, $-10 + 35 + 28 = 53$. Again, supposing the multipliers 10, 5 and 2 to continue, and making $s = 20$ as before, let now $r = 107$, and you will have $c = 67$, $3c = 201$, $y = 1$, $x = 8$, and $z = 11$: so the numbers

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numbers are 8, 1 and 11, which will solve the problem; for $8 \times 10 + 1 \times 5 + 11 \times 2$, that is, $80 + 5 + 22 = 107$. But this problem will also admit of two more affirmative solutions, which according to the fourth observation are 5, 9 and 6; 2, 17 and 1.

E X A M P L E 2.

Let it be required to divide a given number as s into three such parts, that three times one part, a third part of the second, and a fifth part of the last may again make up the number s .

Here the equations are $x + y + z = s$, and $3x + \frac{y}{3} + \frac{z}{5} = s$; but to fit this last equation for use, I reduce it to integral terms by multiplying the whole equation by 15, and so have $45x + 5y + 3z = 15s$; therefore the equations when fitted for use are

$$1^{\text{st}}, \quad x + y + z = s, \text{ and}$$

$$2^{\text{d}}, \quad 45x + 5y + 3z = 15s.$$

Subtract three times the first equation from the second, and you will have $42x + 2y = 12s$; divide the whole equation by 2, because 2 is the greatest common measure of 42 and 2, and you will have $21x + y = 6s$. Here then $a = 21$, $b = 1$, $c = 6s$, and $d = 1$; for the single quotient of a divided by b , or of 21 divided by 1 being rejected for it's singularity, there will be no remaining quotient for continuing the series from 0 and 1; therefore in this case 1 the latter term of the series must be taken for d ; whence cd the dividend will be $6s$, and we shall have the following canon particularly adapted to this problem; I mean, where the multipliers are 3 , $\frac{1}{3}$ and $\frac{1}{5}$, and where the sum of the products is to be the same with the sum of the numbers: divide $6s$ by 21, and the remainder will be

y ; whence we shall have $x = \frac{6s - y}{21}$, and $z = s - x - y$. As for in-

stance, let it be required to divide 100 into three such parts, that three times the first part, a third part of the second, and a fifth part of the third may all together make the same number 100: here $6s = 600$, which being divided by 21 leaves 12 for y ; whence $x = 28$, and $z = 60$: so the numbers 28, 12 and 60 will solve the problem; for $28 \times 3 + \frac{12}{3} + \frac{60}{5}$, that is, $84 + 4 + 12 = 100$. The fourth observation finds also two other affirmative solutions of this problem, to wit, 27, 33, 40; and 26, 54, 20; and these are all; for another operation would sink z to nothing.

E X A M P L E 3.

Let the sum of the products be still equal to the sum of the numbers, and let the multipliers be 3, 1 and $-\frac{1}{3}$, that is, let $3x + y - \frac{z}{3} = s$, or $9x + 3y - z = 3s$, and the equations will now be

Y y

1st,

$$\text{1st, } x + y + z = s.$$

$$\text{2d, } 9x + 3y - z = 3s.$$

Multiply the first equation by -1 , and then subtract it from the second, or (which is the same in effect) add the first equation to the second, and you will have $10x + 4y = 4s$; divide the whole by 2 (the greatest common measure of 10 and 4) and you will have $5x + 2y = 2s$; therefore in this case $a=5$, $b=2$, and $c=2s$. Now the quotients of a continual division from 5 and 2 are 2 and 2; call them 2 and 1, and with them compute the series 0, 1, 2, 3, and you will have $d=3$, and the canon for the multipliers 3, 1 and $-\frac{1}{5}$, where the sum of the products is to be equal to the sum of the numbers, is as follows: divide $6s$

by 5, and the remainder will be y ; then make $\frac{2s-2y}{5} = x$, and $s - x - y = z$. As for example, let $s=20$; then will $6s=120$, which divided by 5 will have nothing or 5 for a remainder; therefore in this case $y=5$, $x=6$, and $z=9$, and the numbers in order are 6, 5 and 9, which will answer the conditions; for first $6+5+9=20$, and secondly $18+5-3=20$. But according to the fourth observation, this problem will admit of two more solutions, which are 4, 10 and 6; and 2, 15 and 3.

N. B. 1st. In the abovewritten canon we made y the remainder of $6s$ divided by 5; but we might have thrown away $5s$, and have made y the remainder of $1s$ divided by 5; and the same reduction may be made in all other cases where the coefficient of s exceeds the divisor.

2dly. This problem includes the 9th, 10th, 11th, 12th, 13th, 14th and 15th, with an infinite number of other particular ones of this kind: but if this solution be applied to them by way of examples, care must be taken to change their notation, and to fix it according to the directions given at the beginning of this solution.

Of the magic square.

PROBLEM 17.

222. Let any odd square number as 49 be proposed whose square root is 7; and let any square figure be divided into 49 lesser square cells, to wit, into 7 ranks of cells, and every rank into 7 cells: It is required to distribute among these cells all the natural numbers from 1 to 49 inclusively, so that the sum of all the numbers in every row, whether taken horizontally, perpendicularly or diagonally may be the same; which figure thus constructed is commonly called a magic square.

SOLU-

SOLUTION.

As 7 is the side of the square proposed, let the seven first digits 1, 2, 3, 4, 5, 6, 7 be represented by the seven first letters of the alphabet *a, b, c, d, e, f, g* respectively; and let these letters be seven times repeated in the same order in seven distinct rows, placed exactly one under another; and let the difference of the rows consist only in this, that every inferior row is to begin with the second term next above it; and they will stand as in the first figure; where the first or uppermost row is *a, b, c, d, e, f, g*; the second *b, c, d, e, f, g, a*; the third *c, d, e, f, g, a, b*; &c. This scheme, which is in form of a square, is called the primitive square, because upon it is founded the construction of the magic square next to be considered.

Figure 1.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>
<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>a</i>
<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>a</i>	<i>b</i>
<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>e</i>	<i>f</i>	<i>g</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>f</i>	<i>g</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>g</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>

Let now any other square divided into forty-nine cells be prepared, as in the second figure, which I shall by way of anticipation call the magic square; and let the blank cells hereof be filled up by the help of the primitive square, in the manner following.

Figure 2.

$7d+d$	$7e+e$	$7f+f$	$7g+g-49$	$7a+a$	$7b+b$	$7c+c$
$7e+c$	$7f+d$	$7g+e-49$	$7a+f$	$7b+g$	$7c+a$	$7d+b$
$7f+b$	$7g+c-49$	$7a+d$	$7b+e$	$7c+f$	$7d+g$	$7e+a$
$7g+a-49$	$7a+b$	$7b+c$	$7c+d$	$7d+e$	$7e+f$	$7f+g$
$7a+g$	$7b+a$	$7c+b$	$7d+c$	$7e+d$	$7f+e$	$7g+f-49$
$7b+f$	$7c+g$	$7d+a$	$7e+b$	$7f+c$	$7g+d-49$	$7a+e$
$7c+e$	$7d+f$	$7e+g$	$7f+a$	$7g+b-49$	$7a+c$	$7b+d$

First, beginning with the first horizontal row of the primitive square, let all the numbers in this row be successively multiplied by 7 the side of the proposed square, proceeding from left to right; and to the several products of this multiplication add in order the numbers of the seventh and lowest horizontal row, and you will have the numbers $7a+g$, $7b+a$, $7c+b$, $7d+c$, $7e+d$, $7f+e$, $7g+f-49$, with which numbers the next horizontal row below the middle one of the magic square is to be filled from left to right, which in the present case happens to be the fifth row, reckoning from the uppermost: but here (as you may see in the figure) 49 must be thrown away from every greater number; for should we receive any number greater than 49 into this square, places would be wanting for others that are less than 49, which the nature of the problem will not allow of.

2dly. Having furnished this row of the magic square, descend all along with your multiplication, and ascend with your addition in the primitive square, so that the row to be multiplied, and the row to be added may always be equally distant from the middle row on each side; that is, multiply all the numbers in the second row of the primitive square by 7, and add to every product a correspondent number in the sixth row, and you will have the numbers $7b+f$, $7c+g$, $7d+a$, &c, for furnishing the sixth row of the magic square.

3dly. In like manner, seven times the third row together with the fifth, will furnish the seventh row of the magic square.

4thly. We come now to the middle row of the primitive square, wherein both the row to be multiplied, and the row to be added meet; therefore this row will of itself alone, without the assistance of any other, furnish out numbers for the first row of the magic square, to wit, $7d+d$, $7e+e$, $7f+f$, &c.

Figure 1.

a	b	c	d	e	f	g
b	c	d	e	f	g	a
c	d	e	f	g	a	b
d	e	f	g	a	b	c
e	f	g	a	b	c	d
f	g	a	b	c	d	e
g	a	b	c	d	e	f

5thly. In like manner seven times the fifth row of the primitive square together with the third, seven times the sixth together with the second, and seven times the lowest together with the highest, will furnish numbers for the second, third and fourth horizontal rows of the magic square. And thus the whole figure will be furnished, at least in letters: my next task therefore must be to demon-

strate that this figure has all the properties of a magic square.

Figure 2.

$7d+d$	$7e+e$	$7f+f$	$7g+g-49$	$7a+a$	$7b+b$	$7c+c$
$7e+c$	$7f+d$	$7g+e-49$	$7a+f$	$7b+g$	$7c+a$	$7d+b$
$7f+b$	$7g+c-49$	$7a+d$	$7b+e$	$7c+f$	$7d+g$	$7e+a$
$7g+a-49$	$7a+b$	$7b+c$	$7c+d$	$7d+e$	$7e+f$	$7f+g$
$7a+g$	$7b+a$	$7c+b$	$7d+c$	$7e+d$	$7f+e$	$7g+f-49$
$7b+f$	$7c+g$	$7d+a$	$7e+b$	$7f+c$	$7g+d-49$	$7a+e$
$7c+e$	$7d+f$	$7e+g$	$7f+a$	$7g+b-49$	$7a+c$	$7b+d$

And first I will demonstrate that it contains all the natural numbers from 1 to 49 inclusively: for whosoever traces the number 7g through all the horizontal rows, will find $7g-49$ combined with all the letters a, b, c, d, e, f, g, though in another order than as they are here placed; which

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which numbers (since $7g - 49 = 0$) are a, b, c, d, e, f, g , that is, all the numbers from 1 to 7 inclusively : in tracing $7a$ after the same manner he will find, when reduced to order, the numbers $7a+a, 7a+b, 7a+c, 7a+d, 7a+e, 7a+f, 7a+g$, which series includes all numbers from 8 to 14 : he will likewise find all the terms of the series $7b+a, 7b+b, 7b+c, 7b+d, 7b+e, 7b+f, 7b+g$, which takes in all the numbers from 15 to 21 : and so on to $7f+g=49$. In the next place, it is plain that the sum of all the horizontal and perpendicular rows will be the same : for whoever considers these rows with any attention, cannot but see, that the same letters are multiplied, and the same added in every one of these rows, only in a different order ; and therefore the sums must all be the same. We come in the last place to consider the two diagonal rows : now if we examine that which descends from left to right, from $7d+d$ to $7b+d$, we shall find the numbers of this first diagonal row to be $7d+d, 7f+d, 7a+d, 7c+d, 7e+d, 7g+d-49, 7b+d$; but $7d+7f+7a+7c+7e+7g+7b=7a+7b+7c+7d+7e+7f+7g$; therefore the sum of the first diagonal row, which is not affected by changing the order of the numbers, will be the same with the sum of these numbers $7a+d, 7b+d, 7c+d, 7d+d, 7e+d, 7f+d, 7g+d-49$; but d is the middle term of the progression a, b, c, d, e, f, g , and therefore the sum of this progression will be the same as if all the terms were d 's, that is, the sum of this progression will be $7d$; if therefore instead of the additious terms in the numbers abovewritten, which are d seven times repeated, we add the terms of the progression a, b, c, d, e, f, g , which will not alter the sum of the whole, we shall have the sum of the first diagonal row equivalent to the sum of the numbers $7a+a, 7b+b, 7c+c, 7d+d, 7e+e, 7f+f, 7g+g-49$, which is the same with the sum of the first horizontal row, and consequently of every other row : in the other diagonal row which ascends from left to right, from $7c+e$ to $7c+c$, we find the number $7c$ seven times repeated, and severally connected with all the letters a, b, c, d, e, f, g , though in a different order ; but $c=d-1$; therefore $7c=7d-7=a+b+c+d+e+f+g-7$; therefore $7c$ seven times repeated equals $7a+7b+7c+7d+7e+7f+7g-49$; therefore the sum of this diagonal row will be the same as if the order and value of it's terms were as follows, to wit, $7a+a, 7b+b, 7c+c, 7d+d, 7e+e, 7f+f, 7g+g-49$, that is, the sum of this diagonal row is the same with that of all the rest. Q. E. D.

Hitherto

Hitherto we have filled our magic square with letters, in order to give a clearer idea of it's nature and composition: but if we would fill our square with numbers, as the problem requires, we must first make a primitive square in numbers, as in figure 3; where the first or uppermost row is 1, 2, 3, 4, 5, 6, 7; the second 2, 3, 4, 5, 6, 7, 1; the third 3, 4, 5, 6, 7, 1, 2; &c: and if by the help of this, a magic square be constructed according to the directions before given, we shall have such a one as is exhibited in the fourth figure.

Figure 3.

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

Figure 4.

32	40	48	7	* 8	16	24
38	46	5	13	21	* 22	30
44	3	11	19	27	35	* 36
* 1	9	17	25	33	41	49
14	* 15	23	31	39	47	6
20	28	* 29	37	45	4	12
26	34	42	* 43	2	10	18

8

Whoever duly considers the composition of the magic square, and the disposition of it's numbers, will be able to draw (or at least to see the evidence of) the following confectaries.

First, that whatever be the side of the square, unity will always stand in the first cell of the middle horizontal row on the left hand: for $7g + a - 49 = 1$; and if we make 7 or g (the side of the square) equal to r , we shall always have for that cell $rr + a - rr = 1$.

2d. If the numbers 1, 2, 3, 4, 5, 6, 7 in the primitive square had been continued *ad infinitum*, the numbers in every horizontal row of the magic square, except where 49 is to be cast away, would have increased from left to right in an arithmetical progression, whose common difference is $8(r+1)$: for in the first row for instance, (and the reason is the same for all the rest,) the products $7d, 7e, 7f, 7g$, &c make an increasing arithmetical progression whose common difference is 7; and the parts to be added, *viz.* d, e, f, g , &c make another increasing progression whose difference is 1; therefore both together, to wit, $7d + d, 7e + e, 7f + f, 7g + g$ &c, will make a progression whose common difference is 8.

3d. But because the numbers in the primitive square are not continued *ad infinitum*, but always break off after 7, returning back again to 1, that is, from g to a , this occasions an irregular term wherever the number a is to be added: I call it so, because it's excess above the term that is before it in the same row will be but unity, whereas according to the law

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law of this progression, it ought to be 8. Of these irregular terms there will be but one in every horizontal row; they are marked in the fourth figure, and beginning with unity are as follows; 1, 15, 29, 43, 8, 22, 36, forming an arithmetical progression whose common difference is 14 ($2r$).

4th. From what has been laid down in the three foregoing consecutaries, and particularly from a due observation of the places of the irregular terms, being taken in the same order as they were given by us in the last, we have a new way of constructing a magic square more expeditious than the former, as being effected without the help of a primitive square, thus: Form from unity an arithmetical progression whose common difference is $2r$, and the number of terms r , which progression in the present case will be 1, 15, 29, 43, 8, 22, 36; of this progression place the first term in the first cell of the middle horizontal row on the left hand of the magic square to be constructed; then descending from thence diagonally to the right, fill the cells you pass through with as many of the other terms of the foregoing progression as you can: these terms in the present case are 15, 29, 43; but if this motion be further continued, the next term 8 will fall without the square under the fifth perpendicular row from the left hand, as you see in the figure: transfer it therefore to the uppermost cell of the same fifth perpendicular row, and then moving downwards diagonally as before, you will find room for the remaining terms of the progression, to wit, 22 and 36. The irregular terms being thus placed in their proper cells, all the other cells will be easily filled thus: suppose I would fill the first horizontal row; I look for the irregular term already placed there, and find it to be 8; from this term 8 I form an arithmetical progression whose common difference is 8, and then moving in the same row towards the right hand, I insert as many of the other terms of this progression as I can, which terms in the present case are 16 and 24; then finding I can move no further this way, I continue the rest of the progression from the other end of the same horizontal row, writing down, from left to right, the numbers 32, 40, 48 and 7: thus the first row will be full; and the rest must be furnished by a like process.

5th. Since in a magic square, the sum of every row, which way soever taken, is the same; if any one has a mind to estimate this sum without actually constructing the figure, he may reason thus: let 7 be the side of the square; then it is plain that in this square will be contained all the natural numbers from 1 to 49 inclusively, which in their natural order make an arithmetical progression, whereof the least term is 1, the greatest 49, and the number of terms 49: add the greatest and least terms together, and the sum will be 50, whose half is 25; therefore 25 is an arithmetic mean between the extremes 1 and 49; and 49 times 25 will be the sum of the whole progression by art.

124; therefore the sum of all the numbers in the whole square, or the sum of the numbers in all the seven horizontal, or in all the seven perpendicular rows, is 49 times 25; therefore the sum of the numbers in any one row is 7 times 25 or 175: and the reasoning is the same in any other case where the side of the square is an odd number. The sum of

all the numbers in any one row is $\frac{r^2 + 1}{2} \times r$; whence I infer, that if $\frac{r^2 + 1}{2}$

be made equal to m , that is, if m be an arithmetic mean between the extremes 1 and r^2 , the sum of every row will be the same as if its cells were all filled with the same number m .

There are many other ways of constructing magic squares both odd and even, as also many other surprizing properties thereto relating, all which I here pass by, because, though this speculation be in itself very curious and entertaining, yet it cannot be denied but that it is of very little or no use in any other parts of the Mathematics. Whoever would see more of these matters, may consult the Memoirs of the Royal Academy of Sciences for the years 1705 and 1710, where he will find this subject almost exhausted by the learned and justly celebrated Mathematicians M. De La Hire and M. Sauveur.



